Option Pricing in HJM Model using an Asymptotic Expansion Method[*][†]

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Abstract

We derived a new analytic approximation formula based on the asymptotic expansion approach for pricing bond options in an HJM model. We also developed a variance reduction method for Monte Carlo simulations utilizing asymptotic expansion, and examined its accuracy and confirmed its validity in a realistic two-factor model.

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1. Introduction

Heath, Jarrow and Morton [1992] presented the general framework of interest rate models which enable the valuation of interest rate derivative securities by automatically reproducing the initial term stricture of interest rates and by specifying the volatility function of instantaneous forward rates (models belonging to this framework are hereinafter referred to as "HJM models").

While a wide range of concrete models in this framework have been studied, it has been difficult to obtain an analytical expression of a standard bond option's or a swaption's price under nonnegative conditions for interest rates. Valuation is thus commonly based on numerical approximation approaches such as Monte Carlo simulation.

In an HJM model, the stochastic differential equation for a (risk neutral) equivalent martingale measure (EMM) of the forward rate for an extremely short interest period commencing when $u(\geq t)$ at time t, denoted by $\{f(t,u)\}_{0\leq t\leq u<\infty}$, is expressed by

$$f(t,u) = f(0,u) + \sum_{i=1}^{n}\int_0^t \sigma_i(f(v,u),v,u)\int_t^u \sigma_i(f(v,z),v,z)dzdv + \sum_{i=1}^{n}\int_0^t \sigma_i(f(v,u),v,u)dW_i(v)$$

Monte Carlo simulation involves simulation based on the discretization of the short-term money market rate $r(t) = f(t,t)$ and the above equation which represents the volatility of all instantaneous forward rates relating to derivative securities subject to valuation, but it takes substantial computing time to achieve a sufficient level of accuracy to withstand practical use. This paper proposes an approach based on asymptotic expansion to overcome this drawback.

The asymptotic expansion approach is an integrated approach to analytic approximations with a sufficient level of accuracy to withstand practical use, including cases in which the price of the underlying assets follows a general (multidimensional) Markov continuous stochastic process when uncertainties in the valuation of European derivative securities are represented by Brownian motion (Kunitomo and Takahashi [1992], Takahashi [1995] and [1999]), and cases in which its price follows a continuous stochastic process that is not necessarily Markovian in relation to interest rates (Kunitomo and Takahashi [2001]). The asymptotic expansion approach may intuitively be referred to as a stochastic Taylor expansion which involves expanding the target stochastic process such that the coefficient of the Brownian motion is around zero, that is, around a nonstochastic process. In mathematical terms, it is legitimized on the basis of Malliavin-Watanabe Calculus in stochastic analysis (for example, refer to Ikeda and Watanabe [1989] and Yoshida [1992]). (For details, refer to Kunitomo and Takahashi [2003a]). Its scope of application in the finance sector is wide ranging, including the valuation of European derivative securities mentioned above, dynamic optimal portfolio (Takahashi and Yoshida [2001a] and [2004], Kobayashi, Takahashi and Tokioka [2001], and Kunitomo and Takahashi [2003b]), and improving the efficiency of Monte Carlo simulation (Takahashi and Yoshida [2001b]). Further, Takahashi and Saito [2003] demonstrated its application to American derivative securities, and Kunitomo and Takahashi [2003b] to jump-diffusion processes. For a full explanation of such applications of the asymptotic expansion approach to the finance sector in general, refer to Kunitomo and Takahashi [2003].

This paper derives an analytic approximation formula for valuating bond options and swaptions, by converting the assumed stochastic differential equation using the so-called forward measure, and by performing asymptotic expansion on the equation. Kunitomo and Takahashi [2001] derived an analytic approximation formula through asymptotic expansion under a risk neutral measure. The approximation formula obtained under the forward measure is much simpler than the original approximation formula, having the advantage of making numerical evaluation easier. Further, Takahashi and Yoshida [2001b] developed a variance reduction method for Monte Carlo simulations utilizing the asymptotic expansion approach. We enhance this approach to show that the variance in Monte Carlo simulations can be reduced even further. The enhanced approach results in much faster convergence in simulations, and makes it possible to reduce analytic approximation errors due to asymptotic expansion. Through numerical calculation, we also reveal the validity of this approach in a two-factor HJM model with a realistic volatility function which satisfies nonnegativity conditions for interest rates.

This paper is structured as follows. In the next section, Section 2, we establish the problem in concrete terms. In Section 3, we derive an analytic approximation formula for pricing bond options based on asymptotic expansion. And finally, in Section 4, we present a variance reduction method for Monte Carlo simulations using asymptotic expansion, and examine the method with an analytic approximation formula based on numeric examples.

2. Problem Establishment

2.1 Bonds as Underlying Assets

Here, we build a model for bond options, the subject of this paper. Firstly, assume that the trading period is $[0,T]$ ($T<\infty$) at the present time 0.

Suppose that "filtered probability space: $(\Omega,F,\{F_t\}_{0\le t\le T},P)$" is given, and that the interest rate and the price of bonds are F_t-measurable stochastic processes. However, assume that filtration meets the usual conditions. Next, fix the interest-bearing bonds—the underlying assets of the options subject to analysis. Assume that the following is given for options.

- Time at which European options are exercised: $\bar{T}$ $(0<\bar{T}<T)$
- Price at which European options are exercised: K

Also assume that the following is given for underlying assets.

- Time at which cash flow is generated: $(\bar{T}<)T_1<\dots<T_m\le T$
- Cash flow generated at each point of time T_j: c_j $(j=1,\dots,m)$

Then, work out the price of options.

2.2 HJM Model

Here, we study the so-called instantaneous forward rate. Suppose that the instantaneous forward rate is an F_t-measurable stochastic process with respect to the given filtered probability space: $(\Omega,F,\{F_t\}_{0\le t\le T},P)$. Assume that when $f(t,u)(t\le u)$, $f(t,u)$, satisfies the following stochastic differential equation.

$$f(t,u)=f(0,u)+\int_0^t\alpha(f(v,u),v,u)dv+\int_0^t\sum_{i=1}^n\sigma_i(f(v,u),v,u)dW_i(v) \qquad (1)$$

However, let

$\alpha(x,y,z),\sigma_i(x,y,z)$ $(1\le i\le n)$ denote a real-valued function with three variables, and

$W(t)=\{W_i(t)\}_{1\le i\le n}$ denote n-dimensional Brownian motion.

In the so-called HJM model proposed by Heath, Jarrow and Morton [1992], the instantaneous forward rate process satisfies the stochastic differential equation in the form of (2.2). The following is a brief summary of this model.

Based on the concept of instantaneous forward rates, if a discount bond matures at time $u(>\bar{T})$, the price process $P(t,u)$ at time t is

$$P(t,u)=\exp\left\{-\int_t^u f(t,z)dz\right\}$$

Therefore, if the discount bond matures at the time at which cash flow is generated for the bond (the underlying asset), denoted by T_j, its price at time t, denoted by $P(t,T_j)$, is expressed by

$$P(t,T_j)=\exp\left\{-\int_t^{T_j} f(t,z)dz\right\} \quad (j=1,\dots,m)$$

Now, suppose that these discount bond prices are divided by the price of a discount bond which matures at $\bar{T}$, and assume that it equals $P^{\bar{T}}(t,T_j)$. Then,

$$P^{\bar{T}}(t,T_j)=\frac{P(t,T_j)}{P(t,\bar{T})}=\exp\left\{-\int_{\bar{T}}^{T_j} f(t,z)dz\right\} \quad (j=1,\dots,m)$$

In an HJM model, if the order of integration is changed by using Formula (1), the following relationship holds.

$$\log P^{\bar{T}}(t,T_j) = -\int_{\bar{T}}^{T_j} f(t,z)dz$$
$$= -\int_{\bar{T}}^{T_j} f(0,z)dz - \int_0^t \int_{\bar{T}}^{T_j} \alpha(f(v,z),v,z)dzdv$$
$$- \int_0^t \int_{\bar{T}}^{T_j} \sum_{i=1}^{n} \sigma_i(f(v,z),v,z)dzdW_i(v)$$

Further, by using Ito's formula, $P^{\bar{T}}(t,T_j)$ satisfies the following stochastic differential equation.

$$dP^{\bar{T}}(t,T_j) = P^{\bar{T}}(t,T_j)\left[d\log P^{\bar{T}}(t,T_j) + \frac{1}{2} d\left\langle \log P^{\bar{T}}(\cdot,T_j)\right\rangle_t \right]$$
$$= P^{\bar{T}}(t,T_j)\left[b(f(t,T_j),t,T_j)dt + \sum_{i=1}^{n} a_i(f(t,T_j),t,T_j)dW_i(t)\right]$$

This, however, assumes that

$$a_i(f(t,T_j),t,T_j) = -\int_{\bar{T}}^{T_j} \sigma_i(f(t,z),t,z)dz$$
$$b(f(t,T_j),t,T_j) = -\int_{\bar{T}}^{T_j} \alpha(f(t,z),t,z)dz + \frac{1}{2}\sum_{i=1}^{n} a_i(f(t,T_j),t,T_j)^2$$

Based on the above, $P^{\bar{T}}(t,T_j)$ is expected to become a martingale by change in measure, provided that the following assumptions are made.

Assumption 1: Using the symbols above, assume the following with respect to trading period $[0,\bar{T}]$ and the respective maturity dates of discount bonds $(\bar{T} <)T_1,...,T_m$.

m×n volatility matrix: $\sigma(t) = \{\sigma(t)\}_{i,j} = a_i(t,T_j)$

m×1 drift vector: $b(t) = (b_j(t)) = b(t,T_j)$

At this time,

(i) $rank(\sigma(t)) = m$ ($a.s$), and

(ii) F_t-measurable n×1 market risk price process vector $\theta(t)$ exists, and the following three conditions are satisfied.

$$\begin{cases} -b(t) + \sigma(t)\theta(t) = \theta \quad (a.s.), \\ \int_0^{\bar{T}} \|\theta(t)\|^2 dt < \infty, \\ \mathrm{E}\left[\exp\left\{ -\int_0^{\bar{T}}\theta(t)dW(t) - \frac{1}{2}\int_0^{\bar{T}} |\theta(t)|^2 dt\right\}\right] = 1 \end{cases}$$

This assumption can be obtained from a no-arbitrage condition. Usually, it is deemed to hold provided that the number of maturity dates of the discount bonds subject to analysis, denoted by m, is greater than the degree of Brownian motion representing the cause of interest rate fluctuations, denoted by n. We proceed with the discussion based on this assumption.

Under this assumption, the original probability measure P and an equivalent probability measure Q exist. Based on probability measure Q, $W^*(t) = (W_i^*(t))$ is an n-dimensional Brownian motion. However,

$$W_i^*(t) = W_i(t) + \int_0^t \theta_i(v)du \qquad 1 \le i \le n$$

Each $P^{\bar{T}}(t,T_j)$ is expressed by

$$dP^{\bar{T}}(t,T_j) = P^{\bar{T}}(t,T_j)\left[\sum_{i=1}^{n} a_i(f(t,T_j),t,T_j)dW_i^*(t)\right]$$

Therefore, each $P^{\bar{T}}(t,T_j)$ is a martingale based on probability measure Q. On the other hand, the drift function must satisfy the following constraint due to Assumption 1.

$$b(f(t,T_j),t,T_j) = \sum_{i=1}^{n} a_i(f(t,T_j),t,T_j)\theta_i(t)$$

Here, the following holds at arbitrary time $u(>\overline{T})$.

$$b(f(t,u),t,u) = -\int_{\overline{T}}^{u}\left\{\alpha(f(t,z),t,z) + \sum_{i=1}^{n} a_i(f(t,z),t,z)\sigma_i(f(t,z),t,z)\right\}dz$$

$$\sum_{i=1}^{n} a_i(f(t,u),t,u)\theta_i(t) = -\int_{\overline{T}}^{u}\left\{\sum_{i=1}^{n}\sigma_i(f(t,z),t,z)\theta_i(t)\right\}dz$$

This is obvious if both sides are differentiated with respect to *u*. It is evident that the constraint of the drift function is satisfied if the following holds with respect to arbitrary time $u(>\overline{T})$.

$$\alpha(f(t,u),t,u) + \sum_{i=1}^{n} a_i(f(t,u),t,u)\sigma_i(f(t,u),t,u) = \sum_{i=1}^{n}\sigma_i(f(t,u),t,u)\theta_i(t)$$

Accordingly, the following results can be obtained when the instantaneous forward rate process is rewritten using Brownian motion based on an alternative measure.

Proposition 1: Suppose that the instantaneous forward rate process $\{f(t,u)\}$ $(0 \le t \le \overline{T}, \overline{T} \le u \le T)$ is given in the form of Stochastic Differential Equation (1). If Assumption 1 holds, the instantaneous forward rate process $\{f(t,u)\}$ is expressed by

$$f(t,u) = f(0,u) + \sum_{i=1}^{n}\int_0^t \sigma_i(f(v,u),v,u)\int_{\overline{T}}^{u}\sigma_i(f(v,z),v,z)dzdv + \sum_{i=1}^{n}\int_0^t \sigma_i(f(v,u),v,u)dW_i^*(v) \qquad (2)$$

provided that $W^*(t) = (W_i^*(t))$ is an n-dimensional Brownian motion based on probability measure *Q*.

As shown above, we acquired a stochastic differential equation which satisfies an instantaneous forward rate in an HJM model based on a no-arbitrage condition, but this stochastic differential equation does not unconditionally have a solution. Therefore, the following assumption is made.

Assumption 2: Assume that volatility function $\sigma_i(f(v,u),v,u)$ consists of real values, is a nonnegative, bounded continuous function with respect to variables $(0 \le v \le \overline{T}, \overline{T} \le u \le T)$, and is Lipschitz-continuous with respect to the first variable. Also assume that the initial instantaneous forward rate $f(0,u)$ is Lipschitz-continuous with respect to variable *u*.

This assumption prevents the divergence of the solution that satisfies Stochastic Differential Equation (2).
For the existence of a unique solution, the following results by Morton [1989] are well-known.

Proposition 2: Based on Assumption 2, there is a unique continuous stochastic process $f(t,u)$ $(0 \le t \le \overline{T}, \overline{T} \le u \le T)$ which satisfies Stochastic Differential Equation (2).

In the following, we study an instantaneous forward rate process which satisfies Stochastic Differential Equation (2) for a volatility function which satisfies this Assumption.

2.3 Option Pricing

Let $V(t)$ $(0 \le t \le \overline{T})$ denote the price of a bond option at time *t* with respect to the above problem establishment. This option may be regarded as a financial derivative product of *m* units of discount bonds which matures at $T_i(j=1,\ldots,m)$. Here, the probability measure *P* and an equivalent probability measure *Q* exist. Each $P^{\overline{T}}(t,T_j)$ is a martingale based on probability measure *Q* according to the above. In other words, if the price of a discount bond maturing at $\overline{T}$ is a numéraire, the price process for *m* discount bonds which mature at $T_i(j=1,\ldots,m)$ are martingales.

Therefore, according to the well-known fact in mathematical finance, if the price of the discount bond maturing at $\overline{T}$ is a numéraire, the option price process $V(t)\ (0 \le t \le \overline{T})$ is a martingale as well based on probability measure Q. Accordingly, the price of the bond option at the present time ($t = 0$), denoted by $V(0)$, becomes

$$\frac{V(0)}{P(0,\overline{T})} = \mathrm{E}^Q\left[\frac{V(\overline{T})}{P(\overline{T},\overline{T})}\right] = \mathrm{E}^Q\left[V(\overline{T})\right]$$
$$\therefore V(0) = P(0,\overline{T})\mathrm{E}^Q\left[V(\overline{T})\right]$$

For example, the call option price can be expressed as follows according to the settings of the underlying assets.

$$\begin{aligned} V_{call}(0) &= P(0,\overline{T})\mathrm{E}^Q\left[\left(\sum_{j=1}^{m} c_j P(\overline{T},\overline{T}_j) - K\right)_+\right] \\ &= P(0,\overline{T})\mathrm{E}^Q\left[\left(\sum_{j=1}^{m} c_j P^{\overline{T}}(\overline{T},\overline{T}_j) - K\right)_+\right] \end{aligned}$$

Similarly, the put option price can be expressed by

$$V_{put}(0) = P(0,\overline{T})\mathrm{E}^Q\left[\left(K - \sum_{j=1}^{m} c_j P^{\overline{T}}(\overline{T},\overline{T}_j)\right)_+\right]$$

However, both assume that $(x)_+ = \max\{x, 0\}$.
In the following, we work out $V_{call}(0)$ and $V_{put}(0)$ based on the instantaneous forward rate process $f(t,u)$ $(0 \le t \le \overline{T}, \overline{T} \le u \le T)$, as written on the basis of Stochastic Differential Equation (2). In pricing, we consider only the situation after the change of measure as stated above.

3. Analytic Approximation of Option Price by Asymptotic Expansion

3.1 Volatility Assumption

From here onwards, we work on pricing using asymptotic expansion. Again, the stochastic differential equation which satisfies an instantaneous forward rate is expressed by

$$\begin{aligned} f(t,u) = f(0,u) &+ \sum_{i=1}^{n}\int_0^t \hat{\sigma}_i(f(v,u),v,u)\int_{\overline{T}}^{u}\hat{\sigma}_i(f(v,z),v,z)dzdv \\ &+ \sum_{i=1}^{n}\int_0^t \hat{\sigma}_i(f(v,u),v,u)dW_i(v) \end{aligned}$$

Although we would like to work out the analytic approximation of option prices by taking an asymptotic expansion approach, we do not provide a strict mathematical explanation or proof of asymptotic expansion in this paper (refer to Kunitomo and Takahashi (2003)). The following is an intuitive explanation of asymptotic expansion.
Firstly, focus on the fact that the volatility function takes on a small value, and substitute the function $\hat{\sigma}_i(x,v,u)$ in the original stochastic differential equation with function $\varepsilon\sigma_i(x,v,u)$ $(0<\varepsilon\le 1)$. The idea is to work out the approximate solution based on the view that it is possible to perform approximation with a polynomial equation of ε that becomes a normal stochastic differential equation if ε is fixed, offers a solution to each ε that changes smoothly with respect to ε, and has a stochastic process as the coefficient in the vicinity of $\varepsilon = 0$. In other words, let

$$f^{(\varepsilon)}(t,u) = f_0(t,u) + \varepsilon f_1(t,u) + \varepsilon^2 f_2(t,u) + o(\varepsilon^2)$$

such that it satisfies

$$\begin{aligned} f^{(\varepsilon)}(t,u) = f(0,u) &+ \varepsilon^2\sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)\int_{\overline{T}}^{u}\sigma_i(f^{(\varepsilon)}(v,z),v,z)dzdv \\ &+ \varepsilon\sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)dW_i(v) \end{aligned}$$

The approximate solution to the original stochastic differential equation can be obtained by working out the stochastic process, which is the coefficient of each ε^k(k=0,1,…) in the asymptotic expansion of $f^{(\varepsilon)}(t,u)$. The following two assumptions are made in order to apply this method.

Assumption 3: (i) Assume that when parameter ε is fixed, volatility $\sigma_i(f^{(\varepsilon)}(v,u),v,u)$ consists of real values, is a nonnegative, bounded continuous function with respect to variables $(0 \le v \le \overline{T}, \overline{T} \le u \le T)$, is smooth with respect to the first function, and all of its derivatives are uniformly bounded with respect to parameter ε.
(ii) Assume that the initial instantaneous forward rate $f(0,u)$ is Lipschitz-continuous with respect to variable u.

Assumption 4: Assume that the following holds with respect to an arbitrary $(0 \le t \le \overline{T}, \overline{T} \le u \le T)$.

$$\sum_{i=1}^{n}\int_0^t \sigma_i\left(f^{(0)}(v,u),v,u\right)^2 dv > 0$$

As explained previously, Morton [1989] guaranteed the existence of a solution to the stochastic differential equation provided that Assumption 3 is made. Further, the following discussion based on the asymptotic expansion approach is mathematically legitimized on the grounds of Assumptions 3 and 4 (refer to Kunitomo and Takahashi [2003]). Accordingly, we proceed with the discussion on the basis that these Assumptions hold.

3.2 Asymptotic Expansion of Instantaneous Forward Rates

Next, perform asymptotic expansion specifically as follows.

$$f^{(\varepsilon)}(t,u) = f_0(t,u) + \varepsilon f_1(t,u) + \varepsilon^2 f_2(t,u) + o\left(\varepsilon^2\right)$$

Then work out each coefficient to satisfy

$$\begin{aligned} f^{(\varepsilon)}(t,u) = f(0,u) &+ \varepsilon^2 \sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)\int_{\overline{T}}^{u}\sigma_i(f^{(\varepsilon)}(v,z),v,z)dzdv \\ &+ \varepsilon\sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)dW_i(v) \end{aligned} \tag{3}$$

As asymptotic expansion can also be executed on the right hand side based on the aforementioned assumption, each coefficient can be calculated by differentiating it up to the number of degrees with respect to ε. Put differently, it is equivalent to performing Taylor expansion on $f^{(\varepsilon)}(t,u)$ in the vicinity of $\varepsilon = 0$. In other words,

$$f^{(\varepsilon)}(t,u) = f^{(\varepsilon)}(t,u)\Big|_{\varepsilon=0} + \varepsilon \frac{\partial f^{(\varepsilon)}(t,u)}{\partial \varepsilon}\Big|_{\varepsilon=0} + \varepsilon^2 \frac{1}{2}\frac{\partial^2 f^{(\varepsilon)}(t,u)}{\partial \varepsilon^2}\Big|_{\varepsilon=0} + o(\varepsilon^2)$$

Firstly, for $f_0(t,u)$, the following is obvious with the substitution of $\varepsilon = 0$.

$$f^{(0)}(t,u) = f(0,u)$$

Next, in order to work out $f_1(t,u)$, differentiate the right hand side with respect to ε and substitute $\varepsilon = 0$, which leads to

$$\begin{aligned} &\frac{\partial}{\partial \varepsilon}\left(\varepsilon^2 \sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)\int_{\overline{T}}^{u}\sigma_i(f^{(\varepsilon)}(v,z),v,z)dzdv\right) \\ &= \varepsilon^2 \frac{\partial}{\partial \varepsilon}\left(\sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)\int_{\overline{T}}^{u}\sigma_i(f^{(\varepsilon)}(v,z),v,z)dzdv\right) \\ &+ 2\varepsilon \sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)\int_{\overline{T}}^{u}\sigma_i(f^{(\varepsilon)}(v,z),v,z)dzdv \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial}{\partial \varepsilon}\left(\varepsilon \sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)dW_i(v)\right) \\ &= \varepsilon \frac{\partial}{\partial \varepsilon}\left(\sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)dW_i(v)\right) + \sum_{i=1}^{n}\int_0^t \sigma_i(f^{(\varepsilon)}(v,u),v,u)dW_i(v) \end{aligned}$$

Thus, $f_1(t,u)$ is expressed by

$$f_1(t,u) = \sum_{i=1}^{n} \int_0^t \sigma_i f^{(\varepsilon)}(t,u) dW_i(v) \Big|_{\varepsilon=0} = \sum_{i=1}^{n} \int_0^t \sigma_i f^{(0)}(t,u) dW_i(v)$$
$$= \sum_{i=1}^{n} \int_0^t \sigma_i (f(0,u), v, u) dW_i(v)$$

For $f_2(t,u)$,

$$\frac{\partial^2}{\partial \varepsilon^2} \left(\varepsilon^2 \sum_{i=1}^{n} \int_0^t \sigma_i (f^{(\varepsilon)}(v,u), v, u) \int_{\bar{T}}^{u} \sigma_i (f^{(\varepsilon)}(v,z), v, z) dz dv \right)$$
$$= \varepsilon^2 \frac{\partial^2}{\partial \varepsilon^2} \left(\sum_{i=1}^{n} \int_0^t \sigma_i (f^{(\varepsilon)}(v,u), v, u) \int_{\bar{T}}^{u} \sigma_i (f^{(\varepsilon)}(v,z), v, z) dz dv \right)$$
$$+ 4\varepsilon \frac{\partial}{\partial \varepsilon} \left(\sum_{i=1}^{n} \int_0^t \sigma_i (f^{(\varepsilon)}(v,u), v, u) \int_{\bar{T}}^{u} \sigma_i (f^{(\varepsilon)}(v,z), v, z) dz dv \right)$$
$$+ 2 \sum_{i=1}^{n} \int_0^t \sigma_i (f^{(\varepsilon)}(v,u), v, u) \int_{\bar{T}}^{u} \sigma_i (f^{(\varepsilon)}(v,z), v, z) dz dv$$

and

$$\frac{\partial^2}{\partial \varepsilon^2} \left(\varepsilon \sum_{i=1}^{n} \int_0^t \sigma_i (f^{(\varepsilon)}(v,u), v, u) dW_i(v) \right)$$
$$= \varepsilon \frac{\partial^2}{\partial \varepsilon^2} \left(\sum_{i=1}^{n} \int_0^t \sigma_i (f^{(\varepsilon)}(v,u), v, u) dW_i(v) \right)$$
$$+ 2 \frac{\partial}{\partial \varepsilon} \left(\sum_{i=1}^{n} \int_0^t \sigma_i (f^{(\varepsilon)}(v,u), v, u) dW_i(v) \right)$$

Therefore, $2\, f_2(t,u)$ is expressed by

$$2 f_2(t,u) = 2 \sum_{i=1}^{n} \int_0^t \sigma_i (f^{(\varepsilon)}(v,u), v, u) \int_{\bar{T}}^{u} \sigma_i (f^{(\varepsilon)}(v,z), v, z) dz dv \Big|_{\varepsilon=0}$$
$$+ 2 \frac{\partial}{\partial \varepsilon} \left(\sum_{i=1}^{n} \int_0^t \sigma_i (f^{(\varepsilon)}(v,u), v, u) dW_i(v) \right) \Big|_{\varepsilon=0}$$
$$= 2 \sum_{i=1}^{n} \int_0^t \sigma_i (f(0,u), v, u) \int_{\bar{T}}^{u} \sigma_i (f(0,z), v, z) dz dv$$
$$+ 2 \sum_{i=1}^{n} \int_0^t \partial \sigma_i (f(0,u), v, u) \left(\sum_{j=1}^{n} \int_0^u \sigma_j (f(0,u), s, u) dW_j(s) \right) dW_i(v)$$

The asymptotic expansion of $f^{(\varepsilon)}(t,u)$ can be calculated up to the second degree based on the above.

Proposition 3: Perform asymptotic expansion up to the second-degree term as follows.

$$f^{(\varepsilon)}(t,u) = f_0(t,u) + \varepsilon f_1(t,u) + \varepsilon^2 f_2(t,u) + o(\varepsilon^2)$$

This becomes

$$
\begin{aligned}
f_0(t,u) &= f^{(0)}(t,u) = f(0,u) \\
f_1(t,u) &= \sum_{i=1}^{n}\int_0^t \sigma_i(f(0,u),v,u)dW_i(v) \\
&= \int_0^t \sigma(f(0,u),v,u)'dW(v) \\
f_2(t,u) &= \sum_{i=1}^{n}\int_0^t \sigma_i(f(0,u),v,u)\int_{\overline{T}}^u \sigma_i(f(0,z),v,z)dzdv \\
&\quad + \sum_{i=1}^{n}\int_0^t \partial\sigma_i(f(0,u),v,u)\left(\sum_{j=1}^{n}\int_0^v \sigma_i(f(0,u),s,u)dW_j(s)\right)dW_i(v) \\
&= \int_0^t \sigma(f(0,u),v,u)'F(v,u)dv \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\left(\int_0^v \partial\sigma_i(f(0,u),v,u)\sigma_j(f(0,u),s,u)dW_j(s)\right)dW_i(v)
\end{aligned}
$$

provided that

$$\partial\sigma_i(x,v,u) = \frac{\partial}{\partial x}\sigma_i(x,v,u)$$

$\sigma(f(0,u),v,u) = n\times 1$ vertical vector of $\sigma_i(f(0,u),v,u)$

$F(v,u) = n\times 1$ vertical vector of $\int_{\overline{T}}^u \sigma(f(0,z),v,z)dz$

$W(\mathrm{t}) = (W_i(\mathrm{t}))$: n-dimensional Brownian motion

( )' denotes matrix transposition.

3.3 Asymptotic Expansion of Bond Price

Next, work out the asymptotic expansion of the bond price at time $\overline{T}$ using the asymptotic expansion of the instantaneous forward rate calculated above. For this purpose, assuming that the discount bond matures at time $u(>\overline{T})$, perform asymptotic expansion on its price at time $\overline{T}$. This becomes

$$
\begin{aligned}
P^{(\varepsilon)}(\overline{T},u) &= \exp\left\{-\int_{\overline{T}}^u f^{(\varepsilon)}(\overline{T},z)dz\right\} \\
&= \exp\left\{-\int_{\overline{T}}^u\left(f_0(\overline{T},z)+\varepsilon f_1(\overline{T},z)+\varepsilon^2 f_2(\overline{T},z)+o(\varepsilon^2)\right)dz\right\} \\
&= \exp\left\{-\int_{\overline{T}}^u f_0(\overline{T},z)dz\right\}\exp\left\{-\int_{\overline{T}}^u\left(\varepsilon f_1(\overline{T},z)+\varepsilon^2 f_2(\overline{T},z)+o(\varepsilon^2)\right)dz\right\} \\
&= \frac{P(0,u)}{P(0,\overline{T})}\left\{1-\varepsilon\int_{\overline{T}}^u f_1(t,z)dz-\varepsilon^2\int_{\overline{T}}^u f_2(t,z)dz+\frac{1}{2}\varepsilon^2\left(\int_{\overline{T}}^u f_1(t,z)dz\right)^2\right\}+o(\varepsilon^2)
\end{aligned}
$$

Under Assumption 3, the order of integration and stochastic integration can be alternated (for example, refer to Ikeda and Watanabe [1989]). Apply this to calculate the above.

$$
\begin{aligned}
&P^{(\varepsilon)}(\overline{T},u) \\
&= \frac{P(0,u)}{P(0,\overline{T})}\left\{1-\varepsilon\int_0^{\overline{T}} F(v,u)'dW(v)+\frac{1}{2}\varepsilon^2\left(\int_0^{\overline{T}} F(v,u)'dW(v)\right)^2\right. \\
&\left.-\varepsilon^2\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{\overline{T}}\int_0^v\left(\int_{\overline{T}}^u \partial\sigma_i(f(0,z),v,z)\sigma_j(f(0,z),s,z)dz\right)dW_j(s)dW_i(v)\right\}+o(\varepsilon^2)
\end{aligned}
$$

Based on this, perform asymptotic expansion on the price of the bond (underlying asset) at time $\overline{T}$.

$$P^{(\varepsilon)}_{m,\{T_j\},\{c_j\}}(\overline{T}) = \sum_{j=1}^{m} c_j P^{(\varepsilon)}(\overline{T},T_j)$$

Then, perform asymptotic expansion.

$$P^{(\varepsilon)}_{m,\{T_j\},\{c_j\}}(\overline{T}) = X_0(\overline{T})+\varepsilon X_1(\overline{T})+\varepsilon^2 X_2(\overline{T})+o(\varepsilon^2)$$

Now, work out each term as shown below.

$$X_0(\overline{T}) = \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})}$$

$$X_1(\overline{T}) = -\int_0^{\overline{T}} \left(\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} F(v,T_j) \right)' dW(v)$$

$$\begin{aligned}
X_2(\overline{T}) = & \frac{1}{2}\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} \left(\int_0^{\overline{T}} F(v,T_j)' dW(v) \right)^2 \\
& - \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} \int_0^{\overline{T}} \int_{\overline{T}}^{T_j} \sigma(f(0,u),v,u)' F(v,u) du dv \\
& - \sum_{i,j=1}^{n} \int_0^{\overline{T}} \left(\int_0^{v} \sum_{k=1}^{m} c_k \frac{P(0,T_k)}{P(0,\overline{T})} \left(\int_{\overline{T}}^{T_k} \alpha_{i,j}(v,s,u) du \right) dW_j(s) \right) dW_i(v) \\
= & \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} \left[\int_0^{\overline{T}} \left(\int_0^{v} F(s,T_j)' dW(s) \right) F(v,T_j)' dW(v) + \frac{1}{2} \int_0^{\overline{T}} \left| F(v,T_j) \right|^2 dv \right] \\
& - \frac{1}{2}\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} \int_0^{\overline{T}} \left| F(v,T_j) \right|^2 dv \\
& - \sum_{i,j=1}^{n} \int_0^{\overline{T}} \left(\int_0^{v} \sum_{k=1}^{m} c_k \frac{P(0,T_k)}{P(0,\overline{T})} \left(\int_{\overline{T}}^{T_k} \alpha_{i,j}(v,s,u) du \right) dW_j(s) \right) dW_i(v) \\
= & \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} \int_0^{\overline{T}} \left(\int_0^{v} F(s,T_j)' dW(s) \right) F(v,T_j)' dW(v) \\
& - \sum_{i,j=1}^{n} \int_0^{\overline{T}} \left(\int_0^{v} \sum_{k=1}^{m} c_k \frac{P(0,T_k)}{P(0,\overline{T})} \left(\int_{\overline{T}}^{T_k} \alpha_{i,j}(v,s,u) du \right) dW_j(s) \right) dW_i(v)
\end{aligned}$$

However, the above assumes that

$$\alpha_{i,j}(v,s,u) = \partial\sigma_i(f(0,u),v,u)\sigma_i(f(0,u),s,u)$$

3.4 Value of Option Maturing at Time $\overline{T}$

Now, use the asymptotic expansion of the bond price calculated above to express the value of the call option maturing at time $\overline{T}$, denoted by $V_{call}^{(\varepsilon)}(\overline{T})$, with respect to exercise price *K*.

$$\begin{aligned}
V_{call}^{(\varepsilon)}(\overline{T}) &= \left(P_{m,\{T_j\},\{c_j\}}^{(\varepsilon)}(\overline{T}) - K \right)_+ \\
&= \left(X_0(\overline{T}) + \varepsilon X_1(\overline{T}) + \varepsilon^2 X_2(\overline{T}) + o(\varepsilon^2) - K \right)_+ \\
&= \left(y + \varepsilon X^{(\varepsilon)}(\overline{T}) \right)_+
\end{aligned}$$

Similarly, express the value of the put option maturing at time $\overline{T}$, denoted by $V_{put}^{(\varepsilon)}(\overline{T})$.

$$\begin{aligned}
V_{put}^{(\varepsilon)}(\overline{T}) &= \left(K - P_{m,\{T_j\},\{c_j\}}^{(\varepsilon)}(\overline{T}) \right)_+ \\
&= \left(K - X_0(\overline{T}) - \varepsilon X_1(\overline{T}) - \varepsilon^2 X_2(\overline{T}) + o(\varepsilon^2) \right)_+ \\
&= \left(-y - \varepsilon X^{(\varepsilon)}(\overline{T}) \right)_+
\end{aligned}$$

However, assume that

$$X^{(\varepsilon)}(\overline{T}) = X_1(\overline{T}) + \varepsilon X_2(\overline{T}) + o(\varepsilon) = \frac{P_{m,\{T_j\},\{c_j\}}^{(\varepsilon)} - X_0(\overline{T})}{\varepsilon}$$

$$y = X_0(\overline{T}) - K$$

From the above, $V_{call}^{(\varepsilon)}(0)$ and $V_{put}^{(\varepsilon)}(0)$ are expressed as follows, using the probability density function of $X^{(\varepsilon)}(\bar{T})$, denoted by $f_\varepsilon(x)$.

$$
\begin{aligned}
V_{call}^{(\varepsilon)}(0) &= P(0,\bar{T})\mathrm{E}\left[\left(y + \varepsilon X^{(\varepsilon)}(\bar{T})\right)_+\right] \\
&= P(0,\bar{T})\int_{-\frac{y}{\varepsilon}}^{\infty}(y + \varepsilon x) f_\varepsilon(x)dx
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
V_{put}^{(\varepsilon)}(0) &= P(0,\bar{T})\mathrm{E}\left[\left(-y - \varepsilon X^{(\varepsilon)}(\bar{T})\right)_+\right] \\
&= P(0,\bar{T})\int_{-\infty}^{-\frac{y}{\varepsilon}}(-y - \varepsilon x) f_\varepsilon(x)dx
\end{aligned} \tag{5}
$$

Ultimately, the approximate value of the option price can be calculated if the probability density function $f_\varepsilon(x)$ can be approximated.

3.5 Asymptotic Expansion of Probability Density Function $f_\varepsilon(x)$

Firstly, perform asymptotic expansion on the characteristic function of $X^{(\varepsilon)}(\bar{T})$, denoted by $\psi(\xi)$.

$$
\begin{aligned}
\psi(\xi) &= \mathrm{E}\left[\exp\left(i\xi X^{(\varepsilon)}(\bar{T})\right)\right] \\
&= \mathrm{E}\left[\exp\left(i\xi X_1(\bar{T})\right)\exp\left(\varepsilon i\xi(X_2(\bar{T})) + o(\varepsilon)\right)\right] \\
&= \mathrm{E}\left[\exp\left(i\xi X_1(\bar{T})\right)\left(1 + \varepsilon i\xi X_2(\bar{T}) + o(\varepsilon)\right)\right] \\
&= \mathrm{E}\left[\exp\left(i\xi X_1(\bar{T})\right)\left(1 + \varepsilon i\xi\left(\mathrm{E}\left[X_2(\bar{T})\middle|X_1(\bar{T})\right]\right)\right)\right] + o(\varepsilon)
\end{aligned}
$$

Then, break this down into $X_2(\bar{T}) = X_{2,1}(\bar{T}) - X_{2,2}(\bar{T})$. However,

$$
\begin{aligned}
X_{2,1}(\bar{T}) &= \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\bar{T})}\int_0^{\bar{T}}\left(\int_0^v F(s,T_j)'dW(s)\right)F(v,T_j)'dW(v) \\
X_{2,2}(\bar{T}) &= \sum_{i,j=1}^{n}\int_0^{\bar{T}}\left(\int_0^v \sum_{k=1}^{m} c_k \frac{P(0,T_k)}{P(0,\bar{T})}\left(\int_{\bar{T}}^{T_k}\alpha_{i,j}(v,s,u)du\right)dW_j(s)\right)dW_i(v)
\end{aligned}
$$

and

$$
\alpha_{i,j}(v,s,u) = \partial\sigma_i(f(0,u),v,u)\sigma_i(f(0,u),s,u)
$$

On the other hand, for $X_1(\bar{T})$,

$$
\begin{aligned}
X_1(\bar{T}) &= -\int_0^{\bar{T}}\left(\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\bar{T})}F(v,T_j)\right)' dW(v) \\
&= \int_0^{\bar{T}}\sigma_{X_1}(v)'dW(v) \quad \left(\sigma_{X_1}(v) = -\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\bar{T})}F(v,T_j)\right)
\end{aligned}
$$

This shows that $X_1(\bar{T})$ follows a normal distribution with an expected value of 0 and variance of Σ. Here,

$$
\Sigma = \int_0^{\bar{T}}\left(\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\bar{T})}F(v,T_j)\right)'\left(\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\bar{T})}F(v,T_j)\right)dv
$$

Focusing on the fact that $X_1(\bar{T})$ is a stochastic integral of a nonstochastic function based on Brownian motion, and on the form of $X_2(\bar{T}) = X_{2,1}(\bar{T}) - X_{2,2}(\bar{T})$, we cite the following proposition for the conditional expected value (for example, lemma 6.4(i) of Kunitomo and Takahashi [2003]).

Proposition 4: Let $W(t) = (W_i(t))$ denote n-dimensional Brownian motion, $q_1(t)$ represent $R \mapsto R^{1\times n}$, and $q_2(t)$ and $q_3(t)$ stand for the nonstochastic functions of $R \mapsto R^{1\times n}$. Suppose that

$$
\Sigma = \int_0^T q_1(t)q_1(t)'dt < \infty
$$

Here, the following relationship holds with respect to arbitrary real number x and arbitrary $t \leq T$.

$$\mathrm{E}\left[\int_0^t \left\{ \int_0^v q_2(s)dW(s) \right\} q_3(v)dW(v) \middle| \int_0^T q_1(t)dW(t) = x \right]$$
$$= \frac{1}{\Sigma^2}(x^2 - \Sigma)\int_0^t \int_0^v q_1(v)q_3(v)' q_2(s) q_1(s)' ds dv$$

By applying this Proposition, the conditional expected values $E[X_{2,1}(\overline{T}) \mid X_1(\overline{T}) = x]$ and $E[X_{2,2}(\overline{T}) \mid X_1(\overline{T}) = x]$ can be calculated, and $E[X_2(\overline{T}) \mid X_1(\overline{T}) = x]$ can be obtained. The calculation result acquired by applying this Proposition is

$$\mathrm{E}\left[X_{2,1}(\overline{T}) \middle| X_1(\overline{T}) = x\right]$$
$$= \frac{1}{2}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})}\left(\int_0^{\overline{T}} F(v,T_j)'\sigma X_1(v)dv\right)^2 \tag{6}$$

$$\mathrm{E}\left[X_{2,2}(\overline{T}) \middle| X_1(\overline{T}) = x\right]$$
$$= \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} \times$$
$$\int_{\overline{T}}^{T_j}\left(\sum_{i,k=1}^{n}\int_0^{\overline{T}} \partial\sigma_1(f(0,u),v,u)\sigma_{X_1}^{(i)}(v)\int_0^v \sigma_k(f(0,u),s,u)\sigma_{X_1}^{(k)}(s)dsdv\right)du$$

provided that $\sigma_{X_1}^{(i)}(t)$ denotes the ith element of $\sigma_{X_1}(t)$.
Therefore, $E[X_2(\overline{T}) \mid X_1(\overline{T}) = x]$ becomes

$$\mathrm{E}\left[X_2(\overline{T}) \middle| X_1(\overline{T}) = x\right] = \frac{c}{\Sigma^2}(x^2 - \Sigma) \tag{7}$$

Here, the following is assigned.

$$c = \frac{1}{2}\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})}\left(\int_0^{\overline{T}} F(v,T_j)'\sigma_{X_1}(v)dv\right)^2$$
$$-\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})}\int_{\overline{T}}^{T_j}\left(\sum_{i,k=1}^{n}\int_0^{\overline{T}} \partial\sigma_i(f(0,u),v,u)\sigma_{X_1}^{(i)}(v)\int_0^v \sigma_k(f(0,u),s,u)\sigma_{X_1}^{(k)}(s)dsdv\right)du$$

Based on the calculation result shown above, the characteristic function of $X^{(\varepsilon)}(\overline{T})$, denoted by $\psi(\xi)$, is approximated to

$$\psi(\xi) = \mathrm{E}\left[\exp\left(i\xi X_1(\overline{T})\right)\left(1 + \varepsilon i\xi\left(\mathrm{E}\left[X_2(\overline{T}) \middle| X_1(\overline{T})\right]\right)\right)\right] + o(\varepsilon)$$
$$= \exp\left\{-\frac{1}{2}\Sigma\xi^2\right\} + \varepsilon\mathrm{E}\left[i\varepsilon\frac{c}{\Sigma^2}\left(X_1(\overline{T})^2 - \Sigma\right)\exp\left(i\xi X_1(\overline{T})\right)\right] + o(\varepsilon)$$
$$= \psi_1^{(\varepsilon)}(\xi) + o(\varepsilon)$$

Suppose that $\psi_1^{(\varepsilon)}(\xi) = E[\exp(i\xi X_1(\overline{T}))(1 + \varepsilon i\xi(E[X_2(\overline{T}) \mid X_1(\overline{T})]))]$.

Here, if the probability density function takes on random variable $Y^{(\varepsilon)}$ such that

$$f_1^{(\varepsilon)}(x) = \left[1 + \varepsilon\left(\frac{c}{\Sigma^3}x^3 - 3\frac{c}{\Sigma^2}x\right)\right] n[x;0,\Sigma]$$

provided that

$$n[x;0,\Sigma] = \frac{1}{\sqrt{2\pi\Sigma}}\exp\left(-\frac{x^2}{2\Sigma}\right)$$

then it is possible to confirm that the characteristic function becomes $\psi_1^{(\xi)}(\xi)$ by performing integration by parts. By this method, we found that the probability density function of $X^{(\varepsilon)}(\overline{T})$, denoted by $f_\varepsilon(x)$, approximates to

$$f_\varepsilon(x) = \left[1 + \varepsilon\left(\frac{c}{\Sigma^3}x^3 - 3\frac{c}{\Sigma^2}x\right)\right] n[x;0,\Sigma] + o(\varepsilon) \tag{8}$$

3.6 Analytic Approximation of Option Price

The analytic approximation of options can be worked out by using the probability density function $f_\varepsilon(x)$ acquired above.

Substituting Formulae (4) and (5) with Formula (8) results in

$$V_{call}^{(\varepsilon)}(0) = P(0,\overline{T})\int_{-\frac{y}{\varepsilon}}^{\infty}(y+\varepsilon x)\left[1 + \varepsilon\left(\frac{c}{\Sigma^3}x^3 - 3\frac{c}{\Sigma^2}x\right)\right] n[x;0,\Sigma]dx + o(\varepsilon)$$

$$V_{put}^{(\varepsilon)}(0) = P(0,\overline{T})\int_{-\infty}^{-\frac{y}{\varepsilon}}(-y-\varepsilon x)\left[1 + \varepsilon\left(\frac{c}{\Sigma^3}x^3 - 3\frac{c}{\Sigma^2}x\right)\right] n[x;0,\Sigma]dx + o(\varepsilon)$$

By repeating integration by parts, the following can be obtained.

$$\begin{aligned}
V_{call}^{(\varepsilon)}(0) &= P(0,\overline{T})\int_{-\frac{y}{\varepsilon}}^{\infty}(y+\varepsilon x) f_\varepsilon(x)dx + o(\varepsilon)\\
&= P(0,\overline{T})\int_{-\frac{y}{\varepsilon}}^{\infty}\left[(y+\varepsilon x) + \varepsilon^2\frac{c}{\Sigma^2}(x^2-\Sigma)\right] n[x;0,\Sigma]dx + o(\varepsilon)\\
&= P(0,\overline{T})\left(\varepsilon\Sigma n\left[\frac{y}{\varepsilon};0,\Sigma\right] + yN\left(\frac{y}{\varepsilon\sqrt{\Sigma}}\right) - \varepsilon\frac{c}{\Sigma}yn\left[\frac{y}{\varepsilon};0,\Sigma\right]\right) + o(\varepsilon)
\end{aligned}$$

$$\begin{aligned}
V_{put}^{(\varepsilon)}(0) &= P(0,\overline{T})\int_{-\infty}^{-\frac{y}{\varepsilon}}(-y-\varepsilon x) f_\varepsilon(x)dx + o(\varepsilon)\\
&= P(0,\overline{T})\int_{-\infty}^{-\frac{y}{\varepsilon}}\left[(-y-\varepsilon x) - \varepsilon^2\frac{c}{\Sigma^2}(x^2-\Sigma)\right] n[x;0,\Sigma]dx + o(\varepsilon)\\
&= P(0,\overline{T})\left(\varepsilon\Sigma n\left[\frac{y}{\varepsilon};0,\Sigma\right] - yN\left(-\frac{y}{\varepsilon\sqrt{\Sigma}}\right) - \varepsilon\frac{c}{\Sigma}yn\left[\frac{y}{\varepsilon};0,\Sigma\right]\right) + o(\varepsilon)
\end{aligned}$$

This boils down to the proposition shown on the next page.

Proposition 5: The approximate solution of the price of European bond options established above is as follows.

$$\begin{aligned}
V_{call}^{(\varepsilon)}(0) &\approx P(0,\overline{T})\int_{-\frac{y}{\varepsilon}}^{\infty}\left(y+\varepsilon x+\varepsilon^2\frac{c}{\Sigma^2}\left(x^2-\Sigma\right)\right) n\left[x;0,\Sigma\right]dx\\
&= P(0,\overline{T})\left(\varepsilon\Sigma n\left[\frac{y}{\varepsilon};0,\Sigma\right] + yN\left(\frac{y}{\varepsilon\sqrt{\Sigma}}\right) - \varepsilon\frac{c}{\Sigma}yn\left[\frac{y}{\varepsilon};0,\Sigma\right]\right)
\end{aligned}$$

$$\begin{aligned}
V_{put}^{(\varepsilon)}(0) &\approx P(0,\overline{T})\int_{-\infty}^{-\frac{y}{\varepsilon}}\left(-y-\varepsilon x-\varepsilon^2\frac{c}{\Sigma^2}\left(x^2-\Sigma\right)\right) n[x;0,\Sigma]dx\\
&= P(0,\overline{T})\left(\varepsilon\Sigma n\left[\frac{y}{\varepsilon};0,\Sigma\right] - yN\left(\frac{y}{\varepsilon\sqrt{\Sigma}}\right) - \varepsilon\frac{c}{\Sigma}yn\left[\frac{y}{\varepsilon};0,\Sigma\right]\right)
\end{aligned}$$

provided that $n[x{:}0\Sigma]$ denotes the following probability density function with a normal distribution, with an expected value of 0 and variance of Σ

$$n[x;0,\Sigma] = \frac{1}{\sqrt{2\pi\Sigma}}\exp\left(-\frac{x^2}{2\Sigma}\right)$$

and $N(x)$ represents the following distribution function with a standard normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt$$

Also, Σ,c,y and other symbols are defined as follows.

$$\Sigma = \int_0^{\overline{T}} \left(\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} F(v,T_j) \right)' \left(\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} F(v,T_j) \right) dv$$

$$c = \frac{1}{2} \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} \left(\int_0^{\overline{T}} F(v,T_j)' \sigma_{X_1}(v) dv \right)^2$$

$$- \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} \int_{\overline{T}}^{T_j} \left(\sum_{i,k=1}^{n} \int_0^{\overline{T}} \partial \sigma_i(f(0,u),v,u) \sigma_{X_1}^{(i)}(v) \int_0^v \sigma_k(f(0,u),s,u) \sigma_{X_1}^{(k)}(s) ds dv \right) du$$

$$y = \sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} - K$$

$\sigma(f(0,z)v,z$: n-dimensional vertical vector whose i th component is $\sigma_I(f(0,z)v,z)$

$$F(v,u) = \int_{\overline{T}}^{u} \sigma(f(0,z),v,z) dz$$

$$\sigma_{X_1}(v) = -\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} F(v,T_j)$$

4. Variance Reduction Method using Asymptotic Expansion

The results of the calculation examples below show that there are some errors in the approximate value acquired through asymptotic expansion when the extent of volatility depends on the level of interest rates. Therefore, we enhance the method advocated by Takahashi and Yoshida [2001] as a way of acquiring the approximate value more accurately, and examine a variance reduction method for standard Monte Carlo simulations using approximation by asymptotic expansion.

As discussed in Section 3,

$$V_{call}^{(\varepsilon)}(0) = P(0,\overline{T}) \mathrm{E}\left[\left(y + \varepsilon X_1(\overline{T}) + \varepsilon^2 X_2(\overline{T}) + o(\varepsilon^2) \right)_+ \right]$$

$$= \mathrm{E}\left[X_2(\overline{T}) \middle| X_1(\overline{T}) = x \right] = \frac{c}{\Sigma^2}(x^2 - \Sigma)$$

Therefore, the right-hand-side of Proposition 5 becomes

$$V_{call}^{(\varepsilon)}(0) \approx P(0,\overline{T}) \mathrm{E}\left[\left\{ y + \varepsilon X_1(\overline{T}) + \varepsilon^2 \mathrm{E}\left[X_2(\overline{T}) \middle| X_1(\overline{T}) \right] \right\} 1_{\{X_1(\overline{T}) \geq -\frac{y}{\varepsilon}\}} \right]$$

$$= P(0,\overline{T}) \mathrm{E}\left[\left\{ y + \varepsilon X_1(\overline{T}) + \varepsilon^2 \frac{c}{\Sigma^2} \left(X_1(\overline{T})^2 - \Sigma \right) \right\} 1_{\{X_1(\overline{T}) \geq -\frac{y}{\varepsilon}\}} \right]$$

Here, $X_1(\overline{T})$ can be expressed by

$$X_1(\overline{T}) = -\int_0^{\overline{T}} \left(\sum_{j=1}^{m} c_j \frac{P(0,T_j)}{P(0,\overline{T})} F(v,T_j) \right)' dW(v)$$

Accordingly, if the statistic for $\overline{V}_{call}^{(\varepsilon)}(\overline{T})$ is

$$\overline{V}_{call}^{(\varepsilon)}(\overline{T}) \equiv \left\{ y + \varepsilon X_1(\overline{T}) + \varepsilon^2 \frac{c}{\Sigma^2} (X_1(\overline{T})^2 - \Sigma) \right\} 1_{\{X_1(\overline{T}) \geq -\frac{y}{\varepsilon}\}}$$

Then the statistic can be calculated by generating $X_1(\overline{T})$ through Monte Carlo simulation.

On the other hand, the statistic representing the value of the option at maturity, which is

$$V_{call}^{(\varepsilon)}(\overline{T}) = (y + \varepsilon X^{(\varepsilon)}(\overline{T})) 1_{\{X^{\varepsilon}(\overline{T}) \geq -\frac{y}{\varepsilon}\}}$$

can also be calculated by generating the following through Monte Carlo simulation.

$$X^{(\varepsilon)}(\overline{T}) = \frac{P^{(\varepsilon)}_{m,\{T_i\},\{c_j\}} - X_0(\overline{T})}{\varepsilon}$$

Statistics $\overline{V}^{(\varepsilon)}_{call}(\overline{T})$ and $V^{(\varepsilon)}_{call}(\overline{T})$ are expected to have a strong correlation with each other. Thus, if new statistic V is defined as

$$V \equiv V^{(\varepsilon)}_{call}(\overline{T}) - \overline{V}^{(\varepsilon)}_{call}(\overline{T}) + \mathrm{E}\left[\overline{V}^{(\varepsilon)}_{call}(\overline{T})\right]$$

the variance of V is expected to be much smaller than that of $V^{(\varepsilon)}_{call}(\overline{T})$ in Monte Carlo simulation. Here, $E[\overline{V}^{(\varepsilon)}_{call}(\overline{T})]$ can be analytically calculated by using the right-hand side of $V^{(\varepsilon)}_{call}(0)$ of Proposition 5. In actual fact, the error in statistic V can be expressed by

$$V - V^{(\varepsilon)}_{call}(0)/P(0,\overline{T}) = \left\{V^{(\varepsilon)}_{call}(\overline{T}) - V^{(\varepsilon)}_{call}(0)/P(0,\overline{T})\right\} - \left\{\overline{V}^{(\varepsilon)}_{call}(\overline{T}) - \mathrm{E}\left[\overline{V}^{(\varepsilon)}_{call}(\overline{T})\right]\right\}$$

The first and second curly brackets on the right-hand side represent the error in $V^{(\varepsilon)}_{call}(\overline{T})$ and $\overline{V}^{(\varepsilon)}_{call}(\overline{T})$, respectively. Therefore, if $V^{(\varepsilon)}_{call}(\overline{T})$ and $\overline{V}^{(\varepsilon)}_{call}(\overline{T})$ have a strong positive correlation with each other, their errors should cancel each other out and the error in V should become smaller. We actually confirmed the validity of this method based on the fact that the standard deviation substantially decreased with the use of statistic V, as shown in the numeric examples below.

5. Accuracy of Asymptotic Expansion in Numeric Examples

5.1 Assumed Model

In this Section, we examine the accuracy of analytic approximation shown in Section 3 by taking swaption as a specific example (swaptions are essentially bond options). We consider only a fixed interest rate (which corresponds to a call option of a bond with exercise price K=1), and fix the maturity term of options to five years, and the underlying assets to a five-year swap, settled once per year.
Next, the volatility function must be decided in concrete terms in order to perform actual calculations. Here, we make the following assumption for the volatility function, so as to express shifts and twists which are claimed to account for more than 90% of all causes of interest rate fluctuations.

Assumption 5: Assume that the Brownian motion representing the causes of interest rate fluctuations is two dimensional, and that the two corresponding volatility functions are as follows.

$\sigma_1^*(x,v,u) = \sigma_1 h(x), \quad \sigma_2^*(x,v,u) = \sigma_2\left(1-\beta\exp^{-a(u-v)}\right)h(x)$

$\sigma_1, \sigma_2, \alpha, \beta$: Nonnegative real numbers

$h(x)$: A bounded, nonnegative, smooth function when $x > 0$

Here, the first and second volatility functions may be deemed to represent the causes of shifts and twists, respectively.
In the following, we calculate a number of volatility functions which satisfy the above assumption in concrete terms. It should be noted that ε in asymptotic expansion does not lose generality even if it is fixed and calculated at ε =1, because adjustments can be made to σ_1 and σ_2. Therefore, we perform calculations by fixing it at ε=1 in the following without explicitly stating so each time.
Also, we consider an initial term structure of interest rates in the form of

$$f(0,u) = 0.03 + 0.004u \qquad \text{(positive straight-line yield)}$$

5.2 Concrete Calculation Examples

Calculation Example 1: Suppose that function $h(x)$ in Assumption 5 is constant, and that the extent of volatility does NOT depend on the level of interest rates.

In this case, the stochastic differential equation which should be satisfied by instantaneous forward rates has an analytic solution, and the distribution of the respective prices of discount bonds is a log-normal distribution. Thus, it is easy to calculate the value of options at maturity $V(\overline{T})$ by computer. Although this does not have to be calculated through asymptotic expansion, we examine the accuracy of asymptotic expansion by actually comparing the results of the two calculation methods. However, the calculation assumes that

$$h(x) = 1 \quad (const), \quad \sigma_1 = 0.2, \quad \sigma_2 = 0.08, \quad \alpha = 0.5, \quad \beta = 2$$

The results (refer to Table 1 below) show that the calculation based on asymptotic expansion is extremely accurate in this case.

Next, we consider a case in which the extent of volatility depends on the level of interest rates. To prepare for this, we provide the following two functions.

Definition 1: The two functions $h_1(x)$ and $h_2(x)$ are defined as

$$h_1(x) = \frac{h_0(10001 - x)}{h_0(x - 10000) + h_0(10001 - x)} \qquad h_2(x) = \frac{h_0(x - 10000)}{h_0(x - 10000) + h_0(10001 - x)}$$

provided that

$$h_0(x) = \begin{cases} \exp^{-\frac{1}{x}} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$$

It is evident that the two functions are both bounded, nonnegative and smooth functions. Consider the following volatility function using these functions.

Calculation Example 2: Suppose that function $h(x)$ in Assumption 5 is

$$h(x) = h_1(x)x^{\gamma} + h_2(x)10000^{\gamma} \qquad (0 \leq \gamma \leq 1)$$

This may be regarded as the smoothly-adjusted version of function $\{\min(x,10000)\}^{\gamma}$. Its volatility type may be considered to be such that the extent of the volatility is proportionate to the level of interest rates to the power of γ.

As the stochastic differential equation which should be satisfied by the instantaneous forward rates cannot be analytically solved in this case, the standard approach is to perform Monte Carlo simulation on the basis of this formula. However, much computing time is consumed in achieving an acceptable level of accuracy for practical use. When the analytic approximation results obtained from the standard approach are compared with those acquired by asymptotic expansion (refer to Tables 2 through 5 below), it shows that analytic approximation is an extremely close approximation.

However, especially in cases where the extent of volatility depends on the level of interest rates, some errors are observed in the approximate value obtained by asymptotic expansion. We therefore applied the method introduced in Section 4 in order to calculate the approximate value with greater accuracy. According to the numerical calculation results (refer to Tables 2 through 5 below), the maximum error rate based on the average value calculated from 1000 paths in 500 cases indicates improvements in accuracy compared to the results of analytic approximation. Further, the standard deviation was lower than standard Monte Carlo simulation: it decreased to less than 10% except in one case (10.78% when $\gamma = 0.25$, 40%OTM), reflecting a substantial reduction in variance.

5.3 Review of Calculation Results

Here, we review the calculation results obtained above. Firstly, the error rate decreased as the fixed interest rate increased from OTM (Out of The Money) to ITM (In The Money) in both cases. Conversely, at 40%OTM, the error rate was maximized, while the absolute level of error was about 3 to 4 base points. Further, when the results with respect to each γ in Calculation Example 2 were compared with those of Calculation Example 1 (corresponding to $\gamma = 0$), we observed a higher error rate as γ increased.

There are two reasons for this: as the distribution of instantaneous forward rates based on the asymptotic expansion approach is a closer approximation to a normal distribution ($\gamma = 0.0$), the accuracy of distribution approximation deteriorates as γ deviates from 0.0; and the change in volatility with respect to interest rate fluctuations is aggravated as γ increases. Consequently, the error rate in price is deemed to be high particularly in the case of OTM, which is affected the most by uneven distribution. However, by applying the aforementioned variance reduction method in both cases, it is possible to substantially reduce the standard deviation of values generated by Monte Carlo simulations, and thus calculate the values with high precision in a shorter period of time.

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● **Notes on Calculation Method**

In Monte Carlo simulation, we generated paths by dividing one year by 365, and performed integration on the instantaneous forward rate by dividing one year by four when calculating the bond price. We generated paths 2.5 million times, and worked out the present value of options by calculating the average value of the corresponding options at maturity. We calculated the coefficients in asymptotic expansion through numerical integration.

For the variance reduction method, we calculated the average value by first generating paths 1000 times. We performed the task of generating paths 1000 times and calculating the average value in 500 cases, and calculated the errors in the average value based on those 500 cases with reference to the average value based on paths generated 2.5 million times in Monte Carlo simulation. We then worked out the standard deviation of the error rate and the maximum error rate with respect to the errors in those 500 cases.

● **Notes on Tables**

Comparison of convergence in the upper row: For Monte Carlo simulation, the figures represent the average value acquired as a result of generating paths 2.5 million times, while the error rate is measured on the basis of the average value of paths generated 2.5 million times in Monte Carlo simulation.

Comparison of error rate in the lower row: We calculated the average value based on paths generated 1000 times in Monte Carlo simulation in 500 cases, calculated the error rate on the basis of the average value based on paths generated 2.5 million times in Monte Carlo simulation, and showed the standard deviation and the maximum error rate regarding those 500 cases.

Table 1: $h(x) = 1$ $(const)$, $\sigma_1 = 0.01$, $\sigma_2 = 0.004$, $\alpha = 0.5$, $\beta = 2$

	40%OTM	20%OTM	10%OTM	ATM	10%ITM	20%ITM	40%ITM
Monte Carlo Simulation	0.006855	0.016660	0.024251	0.033899	0.045653	0.059431	0.092194
Asymptotic Expansion	0.006852	0.016639	0.024230	0.033881	0.045633	0.059408	0.092190
Error Rate	-0.05%	0.13%	0.09%	0.05%	0.04%	0.04%	0.00%

Table 2: $\gamma = 0.25$, $\sigma_1 = 0.02115$, $\sigma_2 = 0.008459$, $\alpha = 0.5$, $\beta = 2$

Comparison of Conversion	40%OTM	20%OTM	10%OTM	ATM	10%ITM	20%ITM	40%ITM
Monte Carlo Simulation	0.006468	0.017023	0.025120	0.035248	0.047374	0.061365	0.094068
Asymptotic Expansion	0.006752	0.017223	0.025284	0.035395	0.047525	0.061537	0.094289
Error Rate	4.38%	1.18%	0.65%	0.42%	0.32%	0.28%	0.23%
Comparison of Error Rate	40%OTM	20%OTM	10%OTM	ATM	10%ITM	20%ITM	40%ITM
Monte Carlo Simulation	—	—	—	—	—	—	—
(1) Standard Deviation	6.58%	3.81%	2.97%	2.26%	1.64%	1.16%	0.58%
Maximum Error Rate	22.86%	13.35%	9.91%	7.47%	5.47%	4.03%	1.94%
Variance Reduction Method	—	—	—	—	—	—	—
(2) Standard Deviation	0.71%	0.33%	0.24%	0.17%	0.11%	0.07%	0.05%
Maximum Error Rate	2.37%	0.91%	0.64%	0.45%	0.35%	0.27%	0.19%
(2)/(1)(%)	10.78%	8.59%	8.18%	7.34%	6.45%	5.88%	7.76%

Table 3: $\gamma = 0.5$, $\sigma_1 = 0.04472$, $\sigma_2 = 0.01789$, $\alpha = 0.5$, $\beta = 2$

Comparison of Conversion	40%OTM	20%OTM	10%OTM	ATM	10%ITM	20%ITM	40%ITM
Monte Carlo Simulation	0.006139	0.017472	0.026097	0.036729	0.049257	0.063506	0.096261
Asymptotic Expansion	0.006557	0.017789	0.026360	0.036969	0.049515	0.063806	0.096648
Error Rate	6.81%	1.82%	1.01%	0.65%	0.52%	0.47%	0.40%
Comparison of Error Rate	40%OTM	20%OTM	10%OTM	ATM	10%ITM	20%ITM	40%ITM
Monte Carlo Simulation	—	—	—	—	—	—	—
(1) Standard Deviation	7.66%	3.99%	3.01%	.23%	1.60%	1.13%	0.58%
Maximum Error Rate	25.32%	13.75%	10.05%	7.37%	5.37%	3.99%	1.94%
Variance Reduction Method	—	—	—	—	—	—	—
(2) Standard Deviation	0.71%	0.27%	0.19%	0.14%	0.10%	0.08%	0.05%
Maximum Error Rate	2.91%	1.00%	0.76%	0.51%	0.39%	0.30%	0.22%
(2)/(1)(%)	9.24%	6.68%	6.21%	6.10%	6.37%	6.82%	8.84%

Table 4: $\gamma = 0.75$, $\sigma_1 = 0.09457$, $\sigma_2 = 0.03783$, $\alpha = 0.5$, $\beta = 2$

Comparison of Conversion	40%OTM	20%OTM	10%OTM	ATM	10%ITM	20%ITM	40%ITM
Monte Carlo Simulation	0.005830	0.017967	0.027144	0.038308	0.051282	0.065843	0.098787
Asymptotic Expansion	0.006262	0.018365	0.027491	0.038641	0.051645	0.066263	0.099307
Error Rate	7.41%	2.22%	1.28%	0.87%	0.71%	0.64%	0.53%
Comparison of Error Rate	40%OTM	20%OTM	10%OTM	ATM	10%ITM	20%ITM	40%ITM
Monte Carlo Simulation	—	—	—	—	—	—	—
(1) Standard Deviation	9.21%	4.25%	3.10%	2.24%	1.59%	1.13%	0.60%
Maximum Error Rate	30.92%	14.40%	10.79%	7.78%	5.41%	4.01%	2.00%
Variance Reduction Method	—	—	—	—	—	—	—
(2) Standard Deviation	0.62%	0.23%	0.17%	0.13%	0.10%	0.06%	0.04%
Maximum Error Rate	2.20%	0.80%	0.56%	0.45%	0.31%	0.25%	0.13%
(2)/(1)(%)	6.70%	5.48%	5.45%	5.79%	6.08%	5.69%	6.60%

Table 5: $\gamma = 1.0$, $\sigma_1 = 0.02$, $\sigma_2 = 0.08$, $\alpha = 0.5$, $\beta = 2$

Comparison of Conversion	40%OTM	20%OTM	10%OTM	ATM	10%ITM	20%ITM	40%ITM
Monte Carlo Simulation	0.005541	0.018523	0.028292	0.040028	0.053489	0.068414	0.101701
Asymptotic Expansion	0.005850	0.018946	0.028678	0.040415	0.053925	0.068925	0.102290
Error Rate	5.57%	2.28%	1.36%	0.96%	0.82%	0.75%	0.58%
Comparison of Error Rate	40%OTM	20%OTM	10%OTM	ATM	10%ITM	20%ITM	40%ITM
Monte Carlo Simulation	—	—	—	—	—	—	—
(1) Standard Deviation	11.27%	4.64%	3.25%	2.30%	1.62%	1.16%	0.63%
Maximum Error Rate	38.59%	15.63%	10.71%	7.67%	5.56%	4.16%	2.23%
Variance Reduction Method	—	—	—	—	—	—	—
(2) Standard Deviation	0.51%	0.26%	0.22%	0.16%	0.12%	0.07%	0.03%
Maximum Error Rate	1.66%	0.70%	0.67%	0.50%	0.33%	0.23%	0.06%
(2)/(1)(%)	4.55%	5.69%	6.77%	6.93%	7.14%	5.92%	4.58%