Term Structure of Interest Rates under Recursive Preferences in Continuous Time

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Abstract

This paper proposes a testable continuous-time term-structure model with recursive utility to investigate structural relationships between the real economy and the term structure of real and nominal interest rates. In a representative-agent model with recursive utility and mean-reverting expectations on real output growth and inflation, this paper shows that, if (1) real short-term interest rates are high during economic booms and (2) the agent is comparatively risk-averse (less risk-averse) relative to time-separable utility, then a real yield curve slopes down (slopes up, respectively). Additionally, for the comparatively risk-averse agent, if (3) expected inflation is negatively correlated with the real output and its expected growth, then a nominal yield curve can slope up, regardless of the slope of the real yield curve.

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1 Introduction

The term structure of interest rates plays a crucial role in practice. From a Macroeconomic perspective, short-term interest rates are a policy instrument conducted by central banks. Bond yield curves imply some market information regarding the future interest rates. Also, from a Finance perspective, fixed-income markets trade a large amount of bonds and derivative securities sensitive to interest rates. The term structure of interest rates is used for pricing not only the bonds and the interest rate derivatives but also all other market securities. In corporate finance, it serves as opportunity costs in investment decisions.

Despite such importance of the term structure of interest rates, surprisingly, people know little about structural relationships between the real economy and the yield curves. Specifically, they know from historical data that, on average, the nominal yield curve slopes up (Homer and Sylla (2005)). Based on standard term structure models such as Cox, Ingersoll, and Ross (CIR, hereafter) (1985b), an upward-sloping yield curve implies that, when the economy is in booms, the *real* short-term rates should be low. However, according to several empirical studies, the GDP growth rates are positively correlated with *nominal* short-term rates. Does the real economy contradict the yield curve?

There is still no agreement about the answer to this question in previous literature. There is a recent growing literature on the term structure of real and nominal interest rates.[1] Notably, Piazzesi and Schneider (2006) predict in a recursive utility model in discrete time that, when inflation is bad news for consumption growth, the nominal yield curve slopes up, whereas the real yield curve slopes down. Empirically, Ang, Bekaert, and Wei (2007) support their results, whereas, by contrast, Hördahl and Tristani (2007) show upward-sloping real and nominal curves.

The purpose of this paper is to provide a tractable, testable framework to answer the question by constructing a continuous-time term structure model in environments with (i) stochastic differential preference, a form of recursive utility, with unitary elasticity of intertemporal substitution and (ii) mean-reverting expectations on inflation and real output. This paper finds that, if (1) real short-term interest rates are high during economic booms and (2) the agent is comparatively risk-averse (less risk-averse) relative to time-separable utility, then a real yield curve slopes down (slopes up, respectively). Additionally, for the comparatively risk-averse agent, if (3) expected inflation is negatively correlated with the real output growth, then a nominal yield curve can slope up, regardless of the slope of the real yield curve.

The main contributions of this paper are twofold. First, this paper is successful in making clear closed-form relationships between the yield curves and structural parameters in continuous time, without resorting to log-linearization approximation. Risk aversion to the uncertainty of future continuation utility, which is specific to time-nonseparable utility, plays a key role, together with the volatility/covariance of underlying state variables, in determining the level and the slope of the real and nominal yield curves. Using these structural parameters and relevant state variables, this model is rich enough to investigate the instantaneous riskless rates and the spot yields in practice.

Especially, some parametric results are consistent with Piazzesi and Schneider (2006)'s prediction, but this paper digs further into this problem in a complementary way to theirs. They predict, using a

[1]For example, see Seppälä (2004).

discrete-time recursive utility model, that, if inflation is bad news for consumption growth, then the nominal yield curve slopes up, while the real yield curve slopes down. Our results shows explicitly the conditions under which our results are consistent with theirs in continuous time.[2] Moreover, our paper probes more deeply into this problem by stressing the effect of the expected inflation shock on the slope of the nominal yield curve. Whereas the inflation shock affects only the level of the nominal yield curve, the negative effect of the endowment growth shock and the expected endowment's growth shock on the expected inflation shock results in the upward-sloping of the nominal yield curve. With regard to role of monetary policy on the term structure, higher credibility in low inflation makes the upward-sloping nominal yield curve flatter.

Second, this paper provides a testable framework to examine the effects of the structural parameters on the yield curves. This framework is applicable directly to the Kalman filtering method. It is useful in examining non-linear characteristics of the yield curve, without resorting to the log-linearization approximation. In addition, this framework enables us to investigate the spot rates and the yield data simultaneously with respect to the same structural parameters. As a by-product, this analysis can expand a data set by using cross-section data on the yield curves.

In related literature, from a financial-engineering viewpoint, this paper uses the method that is explored by Duffie and Epstein (1992a,b), Duffie, Schroder, and Skiadas (1997), Skiadas (1998, 2007), and Schroder and Skiadas (1999). Hence, our technical method itself is not necessarily novel. Rather, by modifying their method, this paper examines the equilibrium yield curves to achieve the closed-form relationships between the real economy and the real and nominal yield curves.

This paper is organized as follows. The next section defines an economy with a recursive utility in continuous time, following Duffie and Epstein (1992a,b), Duffie, Schroder, and Skiadas (1997), Schroder and Skiadas (1999) and Skiadas (2007). Section 3 characterizes the equilibrium real yield curve by setting IES=1 and specifying a mean-reverting endowment process . Section 4 extends the framework into a nominal model by introducing a price index process and a mean-reverting expected inflation process, and characterizes the equilibrium nominal yield curve. Section 5 shows several numerical examples to prove the richness of our framework. The final section concludes. Several supplementary notes and proofs for theorems, propositions and lemmas are placed in Appendices.

2 Economy

Consider a representative-agent endowment economy. Time is $[0, T]$ in continuous time. Let $(\Omega, \{\mathscr{F}_t\}_{0\leq t\leq T}, \mathscr{F}, P)$ denote the filtered probability space that satisfy usual conditions. B denotes an m-dimensional Brownian motion defined on the probability space. There are single non-storable consumption goods. A representative agent is lived on $[0, T]$. She has stochastic differential utility (SDU, henceforth), a form of recursive utility, of consumption as follows:

$$V_t = E_t \left[\int_t^T f(c_u, V_u)\, du \right], \tag{2.1}$$

[2] Our model is rich enough to show other shapes of the term structures by controlling the structural parameters and the conditional variances/covariances of the state variables. See several numerical examples in Section 5.

With regard to the physical environment, define n dimensional Markovian state variables X_t ($n \geq 0$) that satisfy the following a stochastic differential equation (SDE):

$$dX_t = b(X_t, t)dt + a(X_t, t)dB_t, \quad X_0 \in \mathbf{R}^n \tag{2.2}$$

where $b(X, t)$ is an m dimensional vector of drift coefficients and $a(X, t)$ is an $(n \times m)$ matrix of diffusion coefficients. In particular, one element of the state variables is specified as an endowment of the consumption goods. The endowment process is exogenous and is governed by the following stochastic differential equation (SDE):

$$\frac{de_t}{e_t} = \mu_e(X_t, t)dt + \sigma_e(X_t, t)^\top dB_t, \quad e_0 \in \mathbf{R}_+, \tag{2.3}$$

where $\sigma_e(X_t, t)$ is an m dimensional vector of diffusion coefficients. Note that n stands for the number of state variables which relate the endowment. Later, we will specify an additional state variable as an expected endowment growth rate.

Now, we formulate equilibrium pricing, based on Duffie, Schroder and Skiadas (1997). That is, we employ the following pricing kernel:

$$\pi_t = \exp\left\{\int_0^t f_v(e_u, V_u)\, du\right\} f_c(e_t, V_t). \tag{2.4}$$

Using the pricing kernel, the following (real) pricing equation holds: for prices of any securities without dividend, P_t, and time $t < s$,

$$P_t = E_t\left[\frac{\pi_s}{\pi_t} P_s\right]. \tag{2.5}$$

With regard to the spot rate r_t, we have the following lemma with $r_t = -\mathscr{D}\pi_t/\pi_t$ (See Duffie and Epstein (1992b), for example.) where $\mathscr{D}\pi_t$ denotes the drift coefficient of π_t:

Lemma 2.1 *The instantaneous spot rate is obtained as:*

$$r_t = -f_v(e_t, V_t) - \frac{\mathscr{D} f_c(e_t, V_t)}{f_c(e_t, V_t)}. \tag{2.6}$$

Now, we specify the aggregator $f(c_t, V_t)$ in Eq.(2.1) in the stochastic differential utility. To obtain closed-form solutions, we focus on unitary intertemporal elasticity of substitution (IES, henceforth)[3] In particular, we employ three types of the aggregator. They are classified into two categories: time-separable utility and time-nonseparable utility. Under the time-separable utility, we use the following aggregator:

$$f^s(c, v) \triangleq \beta(\log c - v), \tag{2.7}$$

where the superscript s stands for the time separable utility, and β is a positive constant. Using this aggregator, the utility function V_0 becomes:

$$V_0 = \beta E_t\left[\int_0^T e^{-\beta s} u(c_s)\, ds\right]. \tag{2.8}$$

[3]For the stochastic differential utility with general IES, see Appendix A.4.

On the other hand, under the time non-separable utility, we further consider two types of the aggregator as follows:

$$f^{DE}(c,v) \triangleq \beta(1+\alpha v)\left[\log c - \frac{\log(1+\alpha v)}{\alpha}\right] \tag{2.9}$$

$$f^{SS}(c,v) \triangleq (1+\alpha v)\left[\log c - (\beta/\alpha)\log(1+\alpha v)\right]. \tag{2.10}$$

The former aggregator was introduced by Duffie and Epstein (1992a, 1992b) – call this aggregator the *DE aggregator* henceforth. The latter aggregator was explored by Schroder and Skiadas (1999) (hereafter we call it the *SS aggregator*). To obtain the concavity of V_0, the *DE aggregator* is relevant for $\alpha > 0$ whereas the *SS aggregator* is for $\alpha < 0$.[4]

To interpret the coefficient α $(= 1-\gamma)$ intuitively, we take a monotonic transformation of the utility function V_0. For the detailed discussion, see Appendix A.1. Under the *DE aggregator* and the *SS aggregator*, the monotonically transformed utility functions, which are ordinally equivalent to the original utility function V_0, are written as:

$$\tilde{U}_t^{DE} = E_t\left[\int_t^T e^{-\beta(s-t)}\left(\beta\log c_s + \frac{1-\gamma}{2}||\sigma_{U^{DE}}(s)||^2\right)ds\right]. \tag{2.11}$$

$$U_t^{SS} = E_t\left[\int_t^T e^{-\beta(s-t)}\left(\log c_s + \frac{\alpha}{2}||\sigma_{U^{SS}}(s)||^2\right)ds\right]. \tag{2.12}$$

In these functional forms, the time-nonseparable utility can be decomposed into two parts: (1) time-separable utility and (2) additional utility. With respect to the additional-utility, both $||\sigma_{U^{DE}}(s)||^2$ and $||\sigma_{U^{SS}}(s)||^2$ represent the uncertainty of the future continuation utilities. Relative to the part of the time-separable log utility, the additional utility with $\alpha < 0$ (that is, $\gamma - 1 > 0$) results in an additional penalty for the continuation-utility uncertainty, whereas the one with $\alpha > 0$ (that is, $\gamma - 1 < 0$) results in an additional reward for it. Hence, the parts of the additional utility $\frac{1-\gamma}{2}||\sigma_{U^{DE}}(s)||^2$ and $\frac{\alpha}{2}||\sigma_{U^{SS}}(s)||^2$ mean the utility over the uncertainty of the future continuation-utilities. These terms are specific to the time-nonseparable utility. In other words, "$\gamma - 1$ (or, equivalently, $-\alpha$)" means how the agent is risk averse to the uncertainty of the future continuation utility in comparison with the time-separable log utility. This paper calls "$\gamma - 1$ (or, equivalently, $-\alpha$)" a measure of *comparative risk aversion relative to the time-separable log utility.*[5] If $\gamma - 1 > 0$ (or, equivalently, $-\alpha > 0$), then the agent is said to be *comparatively risk-averse relative to the time-separable log utility*, whereas if $\gamma - 1 < 0$ (or, equivalently, $-\alpha < 0$), then the agent is said to be *comparatively less risk-averse relative to the time-separable log utility.* Also, $\gamma = 1$ (that is, $\alpha = 0$) means that the agent pays no attention to the uncertainty of the continuation utility and, thus, that the time-separable log utility function characterizes her utility.

[4] f^{SS} achieves the concavity of V_0 when $\alpha \le \beta$. Since $\beta > 0$, this aggregator is concave not only when $\alpha < 0$ but also when $0 < \alpha \le \beta$. However, since β is very small in practice, the upper limit restricts the positive region of α excessively. Hence, we confine attention to (1) $\alpha < 0$ under the *SS aggregator* and to (2) $\alpha > 0$ under the *DE aggregator*.

[5] The previous literature on SDU (e.g., Schroder and Skiadas (1999, 2007)) calls the coefficient α a measure of preference for time resolution. This paper, by contrast, looks at the negative value of "α (or, equivalently, $1-\gamma$)" and stresses that this new measure means the comparative risk aversion relative to the time-separable log utility. In the following sections, our such interpretation is suitable for drawing the Macroeconomic implications in the SDU model.

3 Real Term Structure

To achieve closed-form solutions, this section focuses two particular forms of the endowment process, which will be specified shortly below. Also, for convenience, this section focuses on the *DE aggregator*. Note that, under the *SS aggregator*, the basic structures are essentially the same as in the *DE aggregator* (See Theorem 4.2). Recall that we assume the unitary IES (i.e., IES=1). Also, to simplify our discussions, we focus on the limit when $T \to \infty$. Appendix provides the result for the case of $T < \infty$.[6]

3.1 Constant mean growth economy

Set $n = m = 1$, that is, there is no other state variable than the endowment shock.

$$\frac{de_t}{e_t} = (\nu - \beta)dt + \sigma_e dB_t, \quad e_0 \in \mathbf{R}_+, \tag{3.1}$$

where $\mu_e \triangleq \nu - \beta$ and B is a Brownian motion under P. Also, ν and σ_e are constants.

With regard to the time-separable utility, we employ the aggregator in Eq.(2.7). With regard to the time-nonseparable utility, use the *DE aggregator* in Eq.(2.9).

Using these two aggregators, we have the following results:

Proposition 3.1 *Suppose the economy with the endowment process Eq.*(3.1) *under the time-separable utility Eq.*(2.7) *and the time-nonseparable utility Eq.*(2.9). *Then the instantaneous riskless rates are as follows:*

$$\begin{aligned} \textit{time-separable case} \quad r^s &= \nu - \sigma_e^2 && (3.2) \\ \textit{time non-separable case} \quad r^{ns} &= r^s + \alpha\sigma_e^2 && (3.3) \end{aligned}$$

where the superscript ns stands for time-nonseparable utility. Since r^s and r^{ns} are constant, the term structures in both case are flat.

Such flat yield curves look unrealistic in general. Therefore, from the next subsection, we confine attention to a stochastic expected endowment growth process.[7]

3.2 Stochastic mean growth economy

Instead of the constant mean growth, we assume that the expected endowment growth process is stochastic, in particular, is mean-reverting (set $n = m = 2$):

$$\begin{aligned} \frac{de_t}{e_t} &= (\nu_t - \beta)dt + \sigma_e^\top dB_t, \quad e_0 \in \mathbf{R}_+, && (3.4) \\ d\nu_t &= k(\bar{\nu} - \nu_t)dt + \sigma_\nu^\top dB_t, \quad \nu_0 \in \mathbf{R} && (3.5) \end{aligned}$$

[6] As is shown in proofs of Lemma A.7 and Lemma A.9 in Appendix, the equilibrium interest rate process depends on T under the time-nonseparable utility.

[7] Stochastic volatility may also affect the shape of the equilibrium term structure of interest rates. Kleshchelski and Vincent (2007) explore an equilibrium yield curve model under stochastic volatility in a robust-control framework. By contrast, this paper focuses on the effect of the stochastic expected endowment growth process on the yield curves in the recursive utility framework.

where $\mu_e \triangleq \nu_t - \beta$. Also, ν_t is stochastic and, in particular, follows the mean-reversion process, whereas $\sigma_e, \sigma_\nu \in \mathbf{R}^m$ and $\bar{\nu}$ is constant. $\bar{\nu}$ denotes the mean-reversion level of the expected endowment growth and k means the speed of the mean reversion.

The following theorem characterizes the relationships among the economic structure, the preference and the term structure.

Theorem 3.1 *Suppose the economy with the endowment process Eq.*(3.4) *and Eq.*(3.5) *under the time-separable utility Eq.*(2.7) *and the time-nonseparable utility Eq.*(2.9). *Then the instantaneous riskless rates and the spot yields are as follows:*

$$\begin{aligned} r_t^s &= \nu_t - ||\sigma_e||^2 && (3.6)\\ r_t^{ns} &= r_t^s + \textit{(Additional endowment shock [DE])} && (3.7)\\ R^s(t,s) &= \textit{(Expectations)} + \textit{(Separable utility's real term premium)} \\ &+ \textit{(Real convexity effect)} && (3.8)\\ R^{ns}(t,s) &= R^s(t,s) + \textit{(Additional endowment shock [DE])} \\ &+ \textit{(Additional expected endowment shock [DE])} && (3.9) \end{aligned}$$

where r_t^s and r_t^{ns} denote the instantaneous riskless rates under the time-separable utility and the time-nonseparable utility respectively, and $R^s(t,s)$ and $R^{ns}(t,s)$ denote the spot yields from time t to s under the time-separable utility and the time-nonseparable utility respectively, and

$$\begin{aligned} TES &\triangleq \left(\sigma_e + \frac{\sigma_\nu}{k+\beta}\right) && (3.10)\\ \textit{(Expectations)} &\triangleq r_t^s + (\bar{\nu} - \nu_t)\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right) && (3.11)\\ \textit{(Separable utility's real term premium)} &\triangleq -\frac{\sigma_\nu^\top \sigma_e}{k}\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right) && (3.12)\\ \textit{(Real convexity effect)} &\triangleq -\frac{||\sigma_\nu||^2}{2k^2}\left(1 - 2\frac{1-e^{-k(s-t)}}{k(s-t)} + \frac{1-e^{-2k(s-t)}}{2k(s-t)}\right) \\ & && (3.13)\\ \textit{(Additional endowment shock [DE])} &\triangleq \alpha\left(||\sigma_e||^2 + \frac{\sigma_e^\top \sigma_\nu}{k+\beta}\right) = \alpha\sigma_e^\top TES && (3.14)\\ \textit{(Additional expected endowment shock [DE])} &\triangleq \frac{\alpha}{k}\left(\sigma_\nu^\top \sigma_e + \frac{||\sigma_\nu||^2}{k+\beta}\right)\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right) \\ &= \alpha\left\{\frac{1}{k}\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right)\sigma_\nu^\top\right\} TES. && (3.15) \end{aligned}$$

Note that, in this paper, *TES* stands for "*Total endowment shock.*" *TES* consists of two parts: (i) σ_e (the shock on the endowment growth) and (ii) σ_ν (the shock on the expected endowment growth). The first part of *TES* is the direct effect of the endowment shock, whereas the second part is the effect of the expectation on the endowment shock multiplied by $\frac{1}{k+\beta}$ (i.e, the reciprocal of the speed of the mean reversion and the time preference). Look at further features of the second part in *TES*. Smaller β results in bigger amplification of the second part, because the agent puts more weight on the future utility. Also, smaller k results in bigger amplification of the second part, because the shock on the expectation of the endowment growth is more persistent. Moreover, note that αTES is a component of the pricing kernel's volatility specific to the time-nonseparable utility.

Examine the detailed effects of the time-nonseparable utility on the real term structure in Theorem 3.1. When we look roughly at the equations in Theorem 3.1, we find that, with regard to the level of the term structure, the term $(-||\sigma_e||^2)$ appears both in the time-separable utility Eq.(3.6) and in the time-nonseparable utility Eq.(3.7), whereas, with regard to the slope of the term structure, both Eq.(3.8) and Eq.(3.9) include Eq.(3.12). On the other hand, the terms Eq.(3.14) and Eq.(3.15) are specific to the time-nonseparable utility.

Now, investigate the effect more closely. With regard to the level of the real term structure, r_t^s and r_t^{ns} have the term $-||\sigma_e||^2$ in common. Since $||\sigma_e||^2 > 0$, this common term pushes down the level of the term structure. On the other hand, r_t^{ns} has the "*Additional endowment shock*" term Eq.(3.14). Suppose that the endowment shock is positively (negatively) correlated with *TES*. Then, for $\alpha > 0$, this term pushes up (down) the level of the term structure (for $\alpha < 0$, vice versa). See also Table 1. The intuitive logic is as follows. As we showed at the end of Section 2, "$-\alpha$ (that is, $\gamma - 1$)" is the measure of the comparative risk aversion relative to the time-separable log utility. Therefore, the agent who has $-\alpha = \gamma - 1 < 0$ is comparatively less risk-averse relative to the time-separable log utility. Suppose that there exists positive (negative) correlation between the endowment shock and *TES*. Then, because the positive (negative) correlation affects the volatility of stochastic differential utility positively (negatively), she receives a positive (negative) reward for such future utility shock through the endowment shock in comparison with the time-separable log-utility when the agent is comparatively less risk-averse relative to the time-separable log-utility, that is $\gamma - 1 = -\alpha < 0$(for $\alpha < 0$, vice versa). As holding the instantaneous short-term bond mitigates this effect, she demands a higher (lower) premium for the bond's yield under the time-nonseparable utility with $\alpha > 0$ than the time-separable utility, which pushes up (down) the level of the real term structure (for $\alpha < 0$, vice versa).

With respect to the slope of the real term structure, $R^s(t, s)$ and $R^{ns}(t, s)$ have the terms Eq.(3.12) and Eq.(3.13) in common. Each term is either increasing or decreasing monotonically in the maturity. First, look at Eq.(3.12). The sign of this term depends on the sign of the correlation between the expected endowment growth shock σ_ν and the endowment growth shock σ_e; if the expected endowment shock is positively (negatively) correlated with the endowment shock, it amplifies (decreases) the uncertainty, and the demand for long-term bonds increases (decreases) due to the risk aversion. Therefore, the slope of the term structure becomes flatter (steeper). Second, the "*Real convexity effect*" term Eq.(3.13) is necessarily negative, because of Jensen's inequality on the expectation operation over the exponential bond discounting under the equilibrium stochastic interest rate process.

Let us draw some intuitive implications of the slope of the real term structure $R^s(t, s)$ from this result from a Macroeconomic perspective. From Eq.(3.6) and Eq.(3.7), the volatility of the expected endowment growth rate σ_ν results, in equilibrium, in the volatility of the spot rates either under the time-separable utility or the time-nonseparable one: the shock on the equilibrium interest rates is equivalent to the shock on the expectation on the GDP growth.[8] Therefore, from a Macroeconomic

[8]Recall that under the above constant mean growth economy, $\sigma_\nu = 0$. Therefore, the equilibrium interest rate is

perspective, the correlation between the endowment growth shock σ_e and the expected endowment growth shock σ_ν means the correlation between the real GDP growth and the real short-term interest rate in equilibrium. As was discussed in the introduction above, according to several empirical studies (e.g., Fama (1990), Orphanides (2002)), interest rates are pro-cyclical: that is, the correlation between the real GDP growth and the *nominal* short-term interest rate is positive. If such positive correlation is also true for real interest rates, the correlation between the endowment growth shock σ_e and the expected endowment growth shock σ_ν is positive. We assume the positive correlation.[9] Hence, Eq.(3.12) is negative and pushes down the slope of the real yield curve. Therefore, as the total of the two terms Eq.(3.12) and Eq.(3.13), under the assumption of the positive correlation of the real GDP and the real short-term interest rates, the real term structure $R^s(t,s)$ slopes down under the time-separable utility. This result is consistent with most previous term-structure models with time-separable utility such as Cox, Ingersoll, and Ross (1985b).

On the other hand, $R^{ns}(t,s)$ involves the "*Additional expected endowment shock*" term Eq.(3.15). This term also varies with the maturity length and influences the slope of the yield in addition to Eq.(3.12) and Eq.(3.13). Let us take a close look at Eq.(3.15). $\left\{\frac{1}{k}\left(1-\frac{1-e^{-k(s-t)}}{k(s-t)}\right)\sigma_\nu\right\}$ stands for the accumulation of the shocks on the expected endowment growth ν_t from t to s. In other words, Eq.(3.15) means that the covariance between the future accumulated shocks on ν_t and *TES* influences the slope of the term structure. For $\alpha > 0$, this term steepens (flattens) the slope of the term structure when the expected endowment shock is positively (negatively) correlated with *TES* (for $\alpha < 0$, vice versa). That is, because the positive (negative) correlation affects the (instantaneous) correlation between the stochastic differential utility and a zero coupon bond's price negatively(positively), holding the bond mitigates the uncertainty of the future utility. Consequently, she receives a negative (positive) reward for holding the bond in comparison with the time-separable log-utility when the agent is comparatively less risk-averse relative to the time-separable log-utility, that is, $\gamma - 1 = -\alpha < 0$(for $\alpha < 0$, vice versa). Hence, she demands a higher (lower) premium for the bond's yield under the time-nonseparable utility with $\alpha > 0$ than the time-separable utility. In addition, those additional effects under the time-nonseparable utility are bigger for the longer-term bonds, regardless of their sign. Thus, the real term structure steepens (flattens) (for $\alpha < 0$, vice versa).

Furthermore, to achieve an intuitive understanding of the effect on $R^{ns}(t,s)$, sort out *TES* into the two parts: σ_e and $\frac{\sigma_\nu}{k+\beta}$. The covariance of the accumulation of the shocks on the expected endowment ν_t (i.e., $\frac{1}{k}\left(1-\frac{1-e^{-k(s-t)}}{k(s-t)}\right)\sigma_\nu$) with the second part $\frac{\sigma_\nu}{k+\beta}$ is necessarily positive because $||\sigma_\nu||^2$ is positive. On the other hand, the effect of the other covariance with the first part σ_e depends on the correlation between the endowment growth shock σ_e and the expected endowment growth shock σ_ν. As was discussed above, from a Macroeconomic perspective, when the correlation between the real GDP growth and the real interest rates is positive, the covariance with the first part is positive. As a whole, when $\alpha > 0$ (i.e., the agent is comparatively less risk-averse relative to the time-separable log utility), the "*Additional expected endowment shock*" term Eq.(3.15) pushes

independent of the bond maturity. Hence, the yield curve is flat in the constant mean growth economy.

[9] This model can also deal with negative correlation between the endowment growth shock σ_e and the expected endowment growth shock σ_ν. This diminishes the total economic growth volatility.

Table 1: Additional endowment shock term, α and correlation

Level	$\alpha > 0$	$\alpha < 0$
$\sigma_e^\top TES > 0$	Up	Down
$\sigma_e^\top TES < 0$	Down	Up

Table 2: Additional expected endowment shock term, α and correlation

Slope	$\alpha > 0$	$\alpha < 0$
$\sigma_\nu^\top TES > 0$	Steepen	Flatten
$\sigma_\nu^\top TES < 0$	Flatten	Steepen

up the slope of the real yield curves under the assumption of the positive correlation of the real GDP and the real short-term interest rates. If the effect is strong enough, the real yield curves can slope up, regardless of the negative effects of the "*Separable utility's real term premium*" term Eq.(3.12) and the "*Real convexity effect*" term Eq.(3.13). On the other hand, when $\alpha < 0$ (i.e., the agent is comparatively risk-averse relative to the time-separable log-utility), the "*Additional expected endowment shock*" term Eq.(3.15) pushes down the slope of the downward-sloping real yield curve further under the same assumption of the positive correlation of the real GDP and the real short-term interest rates. The effects of the additional terms, which are specific to the time-nonseparable utility, are summarized in Table 1 and Table 2.

In short, the macroeconomic factors are explicitly linked with the term structure as follows: Through the covariance with *TES*, σ_e influences r_t^{ns} (i.e., the level of the term structure), whereas σ_ν influences $R^{ns}(t,s)$ (i.e,. the slope of the term structure).

4 Nominal term structure

So far we have examined the real term structure. However, in practice, most fixed income products pay in nominal terms, not in real terms. A real zero coupon bond is a security that pays one unit of consumption goods at its maturity, whereas a nominal zero coupon bond pays one unit of currency at its maturity. This section investigates the nominal term structure by introducing a price index process.

4.1 Nominal term structure

First, set two additional state variables (therefore $n = 2$, $m = 4$): the price index and its expected growth rate (that is, the expected inflation). In particular, the expected inflation process follows a mean-reversion process. Precisely, let N_t denote the price index process and ε is its expected inflation rate as follows:

$$\frac{dN_t}{N_t} = \varepsilon_t dt + \sigma_n^\top dB_t, \quad N_0 \in \mathbf{R}_+ \tag{4.1}$$

$$d\varepsilon_t = \theta(\bar{\varepsilon} - \varepsilon_t)dt + \sigma_\varepsilon^\top dB_t, \quad \varepsilon_0 \in \mathbf{R} \tag{4.2}$$

where θ (the speed of the mean reversion) is a positive constant, $\bar{\varepsilon}$ (the mean-reversion level of the expected inflation rate) is constant, and $\sigma_n, \sigma_\varepsilon \in \mathbf{R}^m$. The processes defined in the previous subsections are modified appropriately.

The pricing equation is as follows: with regard to the nominal price of any asset $\hat{P}_t$,

$$\frac{\hat{P}_t}{N_t} = E_t\left[\frac{\pi_s}{\pi_t}\frac{\hat{P}_s}{N_s}\right]. \tag{4.3}$$

In particular, with regard to the nominal bond that pays one unit of currency at maturity s.

$$\frac{\hat{P}_t}{N_t} = E_t\left[\frac{\pi_s}{\pi_t}\frac{1}{N_s}\right]. \tag{4.4}$$

Note that, for any variable x in real terms, $\hat{x}$ denotes the nominal value of x.

Using Eq.(4.4), look at the role of the inflation factors (that is, the price index process N and the expected inflation process ε) in the equilibrium pricing. Decompose the right hand side of the equilibrium pricing formula Eq.(4.4) into two parts: the real pricing kernel $\frac{\pi_s}{\pi_t}$ and the real payoff at the maturity $\frac{1}{N_s}$. This model implicitly assumes that the agent maximizes her utility of real consumption, not of nominal one. Hence, the optimal utility J remains the same as the one in the previous real economy, except that the number of the state variables is increased. In particular, the real pricing kernel $\frac{\pi_s}{\pi_t}$ is the same as the one in the previous real economy, being independent of the inflation factors (the price index process N and the expected inflation process ε).[10] In other words, the inflation factors influence the equilibrium price only through the real payoff, not through the real pricing kernel. In particular, a higher level of the price index depreciates the real value of the nominal payoff. When the agent is risk averse, she puts more weight on bad states in pricing the bond. Hence, when the inflation factors covariate more positively with the real pricing kernel, the price (the premium) of the nominal bond declines (increases).

Now, obtain some results of the nominal term structure of interest rates under the time-separable utility and the time-nonseparable one.

Theorem 4.1 *Suppose the nominal economy with the endowment process Eq.*(3.4) *and Eq.*(3.5) *and the inflation factors Eq.*(4.1) *and Eq.*(4.2) *under the time-separable utility Eq.*(2.7) *and the time-nonseparable utility Eq.*(2.9)*. Then the nominal instantaneous riskless rates and the nominal spot*

[10]Suppose, on the contrary to our model, that the agent maximizes her utility of *nominal* consumption. Then, the optimal utility J is affected by the price index N; the inflation factors can influence the real pricing kernel.

yields are as follows:

$$\begin{aligned}
\hat{r}_t^s &= r_t^s + \varepsilon_t + \textit{(Risk aversion [level])} && (4.5)\\
\hat{r}_t^{ns} &= \hat{r}_t^s + \textit{(Additional endowment shock [DE])}\\
&+ \textit{(Additional inflation shock [DE])} && (4.6)\\
\hat{R}^s(t,s) &= R^s(t,s) + \textit{(Expected inflation rate)} + \textit{(Risk aversion [level])}\\
&+ \textit{(Separable utility's nominal term premium)}\\
&+ \textit{(Nominal convexity effect)} && (4.7)\\
\hat{R}^{ns}(t,s) &= \hat{R}^s(t,s) + \textit{(Additional endowment shock [DE])}\\
&+ \textit{(Additional inflation shock [DE])}\\
&+ \textit{(Additional expected endowment shock [DE])}\\
&+ \textit{(Additional expected inflation shock [DE])} && (4.8)
\end{aligned}$$

where $\hat{r}_t^s$ and $\hat{r}_t^{ns}$ denote the nominal instantaneous riskless rates under the time-separable utility and the time-nonseparable utility respectively, and $\hat{R}^s(t,s)$ and $\hat{R}^{ns}(t,s)$ denote the nominal spot yields from time t to s under the time-separable utility and the time-nonseparable utility respectively, and

$$\begin{aligned}
\textit{(Expected inflation rate)} &\triangleq E_t\left[\frac{1}{s-t}\log\frac{N_s}{N_t}\right] + \frac{||\sigma_n||^2}{2} && (4.9)\\
\textit{(Risk aversion [level])} &\triangleq -||\sigma_n||^2 - \sigma_n^\top \sigma_e && (4.10)\\
\textit{(Separable utility's nominal term premium)} &\triangleq -\frac{\sigma_\varepsilon^\top \sigma_e}{\theta}\left(1 - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right)\\
&- \left[\frac{\sigma_n^\top \sigma_\varepsilon}{\theta}\left(1 - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right) + \frac{\sigma_n^\top \sigma_\nu}{k}\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right)\right]\\
& && (4.11)\\
\textit{(Nominal convexity effect)} &\triangleq -\frac{||\sigma_\varepsilon||^2}{2\theta^2}\left(1 - 2\frac{1-e^{-\theta(s-t)}}{\theta(s-t)} + \frac{1-e^{-2\theta(s-t)}}{2\theta(s-t)}\right)\\
&- \frac{\sigma_\varepsilon^\top \sigma_\nu}{k\theta}\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)} - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right.\\
&\qquad \left. + \frac{1-e^{-k(s-t)}}{k(s-t)}\frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right) && (4.12)\\
\textit{(Additional inflation shock [DE])} &\triangleq \alpha\left(\sigma_n^\top \sigma_e + \frac{\sigma_n^\top \sigma_\nu}{k+\beta}\right) = \alpha\sigma_n^\top TES && (4.13)\\
\textit{(Additional expected inflation shock [DE])} &\triangleq \frac{\alpha}{\theta}\left(\sigma_\varepsilon^\top \sigma_e + \frac{\sigma_\varepsilon^\top \sigma_\nu}{k+\beta}\right)\left(1 - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right)\\
&= \alpha\left\{\frac{1}{\theta}\left(1 - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right)\sigma_\varepsilon^\top\right\} TES. && (4.14)
\end{aligned}$$

The basic structure remains essentially the same as in real terms, except that several additional terms appear due to the additional state variables (that is, inflation factors). The terms defined in Eq.(4.13) and Eq.(4.14) are specific to the nominal term structure under the time-nonseparable utility. The proof of this theorem follows the same procedures as in Theorem 3.1, and thus is omitted here.[11]

With regard to the level of the nominal term structure (Eq.(4.6)), the "*Additional inflation shock*" term Eq.(4.13) arises through the covariance between the inflation shock σ_n and *TES* multiplied by α. By sorting out *TES* into σ_e and $\frac{\sigma_\nu}{k+\beta}$, we find that the level of the nominal yield curve is affected, relative to the level of the real yield curve, by two sources: (i) the covariance between the endowment growth and the inflation and (ii) the covariance between the expected endowment growth and the inflation. Thus, for $\alpha > 0$, this term pushes up (down) the level of the nominal term structure, relative to the real one, when the inflation shock is positively (negatively) correlated with the total of (i) the endowment growth shock and (ii) the expected endowment growth shock (for $\alpha < 0$, vice versa).

With regard to the slope of the nominal term structure (Eq.(4.8)), the expected inflation shock produces the additional term Eq.(4.14) through the covariance between the expected inflation shock and *TES*. Again, by sorting out *TES* into σ_e and $\frac{\sigma_\nu}{k+\beta}$, we find that the slope of the nominal yield curve is affected, relative to the slope of the real yield curve, by two sources: (i) the covariance between the endowment growth and the expected inflation and (ii) the covariance between the expected endowment growth and the expected inflation. For $\alpha > 0$, this term steepens (flattens) the slope of the nominal term structure, relative to the real one, when the expected inflation shock is positively (negatively) correlated with the total of (i) the endowment growth shock and (ii) the expected endowment growth shock multiplied by $\frac{1}{k+\beta}$ (for $\alpha < 0$, vice versa).

Finally, we state briefly the results under the *SS aggregator*:[12]

Theorem 4.2 *When the representative agent has the SS aggregator, Theorem 3.1 and Theorem 4.1 hold by replacing the additional shock terms with:*

$$\textit{(Additional endowment shock [SS])} \triangleq \alpha\left(\frac{||\sigma_e||^2}{\beta} + \frac{\sigma_e^\top \sigma_\nu}{k+\beta}\right) \tag{4.15}$$

$$\textit{(Additional expected endowment shock [SS])} \triangleq \frac{\alpha}{k}\left(\frac{\sigma_\nu^\top \sigma_e}{\beta} + \frac{||\sigma_\nu||^2}{k+\beta}\right)\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right) \tag{4.16}$$

$$\textit{(Additional inflation shock [SS])} \triangleq \alpha\left(\frac{\sigma_n^\top \sigma_e}{\beta} + \frac{\sigma_n^\top \sigma_\nu}{k+\beta}\right) \tag{4.17}$$

$$\textit{(Additional expected inflation shock [SS])} \triangleq \frac{\alpha}{\theta}\left(\frac{\sigma_\varepsilon^\top \sigma_e}{\beta} + \frac{\sigma_\varepsilon^\top \sigma_\nu}{k+\beta}\right)\left(1 - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right) \tag{4.18}$$

In the next subsection, by imposing several relevant parametric assumptions, we draw several Macroeconomic implications of real and nominal yield curves in relationships to the structural variables.

[11] The proof is available on request from the authors.

[12] For the details, see Schroder and Skiadas (1999).

Table 3: Additional inflation shock term, α and correlation

Level	$\alpha > 0$	$\alpha < 0$
$\sigma_n^\top TES > 0$	Up	Down
$\sigma_n^\top TES < 0$	Down	Up

Table 4: Additional expected inflation shock term, α and correlation

Slope	$\alpha > 0$	$\alpha < 0$
$\sigma_\varepsilon^\top TES > 0$	Steepen	Flatten
$\sigma_\varepsilon^\top TES < 0$	Flatten	Steepen

4.2 Macroeconomic implications on real and nominal yield curves: Some insights into monetary policy

This subsection draws several Macroeconomic implications of real and nominal yield curves in relationships to the structural variables from this model, using the results obtained in the previous two subsections. In particular, we focus on the slope of the yields. In addition, we provide some insights into monetary policy implications.

According to most previous literature on the equity premium puzzle and the risk free rate puzzle, the representative agent is more risk averse preference than log utility. Hence, we assume that $\alpha < 0$. In that case, suppose also that the real yield curve slope down when $\sigma_\nu^\top TES > 0$ from the discussions in Subsection 3.2.

On the other hand, with respect to the level and slope of the nominal yield curve, recall that, as we discussed in the previous subsection, we can interpret that the nominal factors affect the equilibrium nominal price only through the real payoff, not through the real pricing kernel. In particular, suppose that inflation factors(the price index and the expected inflation) have negative covariances with TES (i.e. $\sigma_n^\top(\sigma_e + \frac{\sigma_\nu}{k+\beta}) < 0$ and $\sigma_\varepsilon^\top(\sigma_e + \frac{\sigma_\nu}{k+\beta}) < 0$). Then, when the representative agent is comparatively risk-averse relative to the time-separable log-utility (i.e., $\alpha < 0$), the real pricing kernel covariates more positively with the inflation factors than the time-separable log-utility agent's one. Therefore, she demands less nominal bonds, and thus demands a higher premium for the nominal bonds.

In addition, since longer maturity causes larger covariation between the real pricing kernel and the expected inflation factor, she demands a higher premium for the longer-term nominal bonds. Therefore, the nominal yield curve can slope up, even though the real yield curve slopes down.

Also, when the negative effect of the endowment growth shock on the expectation shock of inflation is not strong enough, the nominal yield curve can slope up in a short-maturity region, while it can slope down in longer-maturity regions, because the convexity effect of the bond discounting is getting stronger by square scale in the longer maturity regions and overcomes the pushing-up effect of the "*Additional expected inflation shock*." This case results in a hump-shaped nominal yield curve.

Furthermore, this model can draw monetary policy implications by confining more attention to the effects of (i) the expected inflation shock and (ii) the speed of the mean reversion on the slope of the nominal yield curve. Consider the case that the nominal yield curve is upward-sloping. Note, as was discussed above, that this case occurs when $\sigma_\varepsilon^\top \sigma_e + \frac{\sigma_\varepsilon^\top \sigma_\nu}{k+\beta} < 0$. Suppose that a central bank can control inflation expectation properly (for example, by achieving some reputation as a strict inflation fighter and, possibly, by setting inflation targeting). Then, the economy achieves lower volatility of the expected inflation. This case results in a flatter nominal yield curve and improves the central bank's controllability over longer-term interest rates in parallel by controlling short rates. Also, when the shock of the expected inflation is less persistent in the economy (that is, when θ is higher), the nominal term structure becomes flatter. In consequence, higher credibility in low inflation makes the upward-sloping term structure of interest rates flatter.

In relationship to previous literature, the paper of Piazzesi and Schroder (2006) is close to ours. They predict, using a discrete-time recursive utility model, that, if inflation is bad news for consumption growth, then the nominal yield curve slopes up, while the real yield curve slopes down. On the other hand, our paper shows that (1) the downward-sloping real yield curve comes from the comparative risk aversion relative to the time-separable log utility and the positive correlation between the endowment growth and the real interest rates and (2) the negative correlation between the inflation factors and the endowment growth can cause the upward-sloping nominal yield curve, regardless of the downward-sloping real one. These parametric results are consistent with Piazzesi and Schroder (2006). Still, this paper digs more deeply into this problem in a tractable continuous-time framework by stressing the effect of the expected inflation shock on the slope of the nominal yield curve. Whereas the inflation shock affects only the level of the nominal yield curve, the negative effect of the endowment growth shock on the expected inflation shock can result in the upward-sloping of the nominal yield curve.

5 Numerical Examples

Based on the qualitative results in Theorem 3.1 and Theorem 4.1, we implement several numerical examples to compare the real and nominal term structures under the time-non-separable utility with those under the time-separable utility, using relevant parametric assumptions.

First, we examine the case of $\alpha > 0$. The results are reported in Figure 1. Table 5 summarizes the parameters that are used in these numerical examples. For simplicity, set $\bar{\nu} = \nu_0$ and $\bar{\varepsilon} = \varepsilon_0$. In addition, take $\rho_{\varepsilon n} = \rho_{en} = \rho_{\nu n} = 0$ to concentrate on the analysis of the effects on the slope, not on the level, of the nominal term structure. Now, look at the case of $\rho_{e\nu} > 0$, $\rho_{e\varepsilon} > 0$ and $\rho_{\nu\varepsilon} > 0$. As shown in Table 1 and Table 2, $\alpha > 0$ and $\rho_{e\nu} > 0$ imply that the "Additional endowment shock" term Eq. (3.14) that is specific to the time-non-separable utility pushes up the level of the real term structure and "Additional expected endowment shock" term Eq. (3.15) that is also specific to the time-non-separable utility steepens it. Hence this case results in the upward-sloping of the real term structure under the time-non-separable utility, whereas the real term structure is downward-sloping under the time-separable utility. Furthermore, $\rho_{e\varepsilon} > 0$ and $\rho_{\nu\varepsilon} > 0$ imply $\sigma_\varepsilon^\top TES > 0$. As shown in Table 4, when $\alpha > 0$, the nominal term structure becomes upward-sloping due to $\sigma_\varepsilon^\top TES > 0$. Figure 1 shows that, when $\rho_{e\nu} > 0$ (i.e, when the interest rates are high in economic booms), the time-non-separable utility model can generate an upward-sloping nominal yield curve, whereas the time-separable utility is not.

Second, we examine the case of $\alpha < 0$. Figure 2 reports the term structures generated by the SS aggregator. Table 6 summarizes the parameters that are used in these examples. In addition to the negative α, we change the values of $\rho_{e\varepsilon}$ and $\rho_{\nu\varepsilon}$ as well in comparison with the above positive α case: $\rho_{e\varepsilon} < 0$ and $\rho_{\nu\varepsilon} < 0$. Under these parametric structures, when $\alpha < 0$ and $\rho_{e\nu} > 0$, the term that is specific to the non-separable utility pushes down the level of the real term structure (Table 1) and flattens it (Table 2). Thus this case results in the downward-sloping of the real term structure. On the other hand, since $\rho_{e\varepsilon} < 0$ and $\rho_{\nu\varepsilon} < 0$ imply $\sigma_\varepsilon^\top TES = \sigma_\varepsilon^\top \sigma_e + \dfrac{\sigma_\varepsilon^\top \sigma_\nu}{k+\beta} < 0$, the nominal term structure slopes up when $\alpha < 0$ (Table 4).

Table 5: Parameter set for positive α case

α	0.8	θ	0.5
β	1%	$\bar{\varepsilon}$	3%
k	0.1	ε_0	3%
$\bar{\nu}$	3%	$\|\|\sigma_\varepsilon\|\|$	1.0%
ν_0	3%	$\|\|\sigma_n\|\|$	5.0%
$\|\|\sigma_\nu\|\|$	2.0%	$\rho_{\varepsilon n}$	0%
$\|\|\sigma_e\|\|$	5.0%	ρ_{en}	0%
$\rho_{e\nu}$	50%	$\rho_{\nu n}$	0%
		$\rho_{e\varepsilon}$	50%
		$\rho_{\nu\varepsilon}$	50%

Figure 1: Positive α

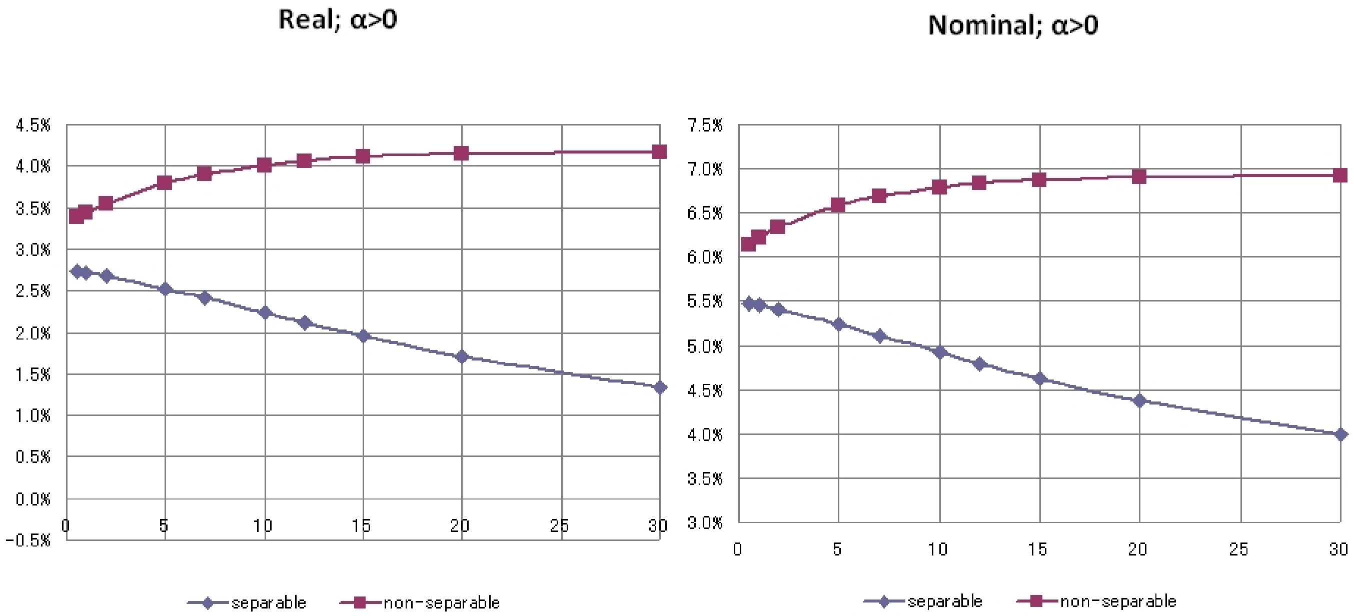
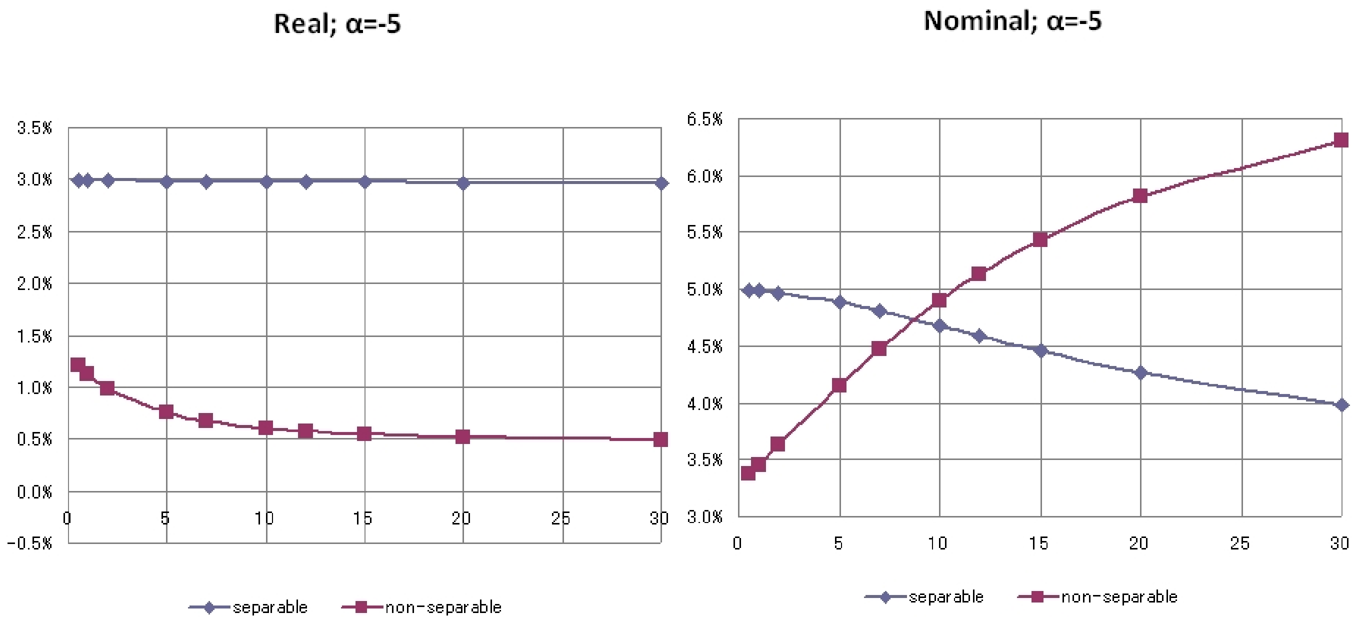


Table 6: Parameter set for negative α case

α	-5	θ	0.1
β	3%	$\bar{\varepsilon}$	3%
k	0.5	ε_0	3%
$\bar{\nu}$	3%	$\|\|\sigma_\varepsilon\|\|$	2.0%
ν_0	3%	$\|\|\sigma_n\|\|$	10.0%
$\|\|\sigma_\nu\|\|$	1.0%	$\rho_{\varepsilon n}$	0%
$\|\|\sigma_e\|\|$	1.0%	ρ_{en}	0%
$\rho_{e\nu}$	20%	$\rho_{\nu n}$	0%
		$\rho_{e\varepsilon}$	-20%
		$\rho_{\nu\varepsilon}$	-20%

Figure 2: Negative α



6 Conclusion

This paper proposes a testable continuous-time term structure model with recursive utility to investigate the structural relationships between the real economy and the term structure of real and nominal interest rates. The main results of this paper are summarized in Theorem 3.1 and Theorem 4.1, both of which provide the crucial effects of the non-separable utility. The numerical examples show the richness of our model in comparison with the classical time-separable utility models and the discrete-time recursive utility models, without resorting to log-linearization approximation methods. The richness of our model provides some advantages in empirical research.

For further research, two points are important. First, empirical research with Kalman filtering enables us to study some direct application of this model to actual financial data. Second, we should deal, in our continuous-time framework, with the IES that is different from unity. This paper restricts attention to the unitary IES to achieve the closed-form solutions in continuous time. On the other hand, according to several discrete-time papers (e.g., Bansal and Yaron (2004)), IES may not be equal to one. The study of such general IES in a continuous-time setting may lead to more general results.

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A Supplementary notes and proofs of theorems

A.1 Note on stochastic differential utility

This note reviews a stochastic differential utility, a form of recursive utility in continuous time, based mainly on Duffie and Epstein (1992a,1992b) and Skiadas (2007). Suppose that a stochastic differential utility follows the process:

$$dU_t = -\left\{ g(c_t, U_t) - \frac{1}{2} A(U_t) ||\sigma_U(t)||^2 \right\} dt + \sigma_U(t) dB_t, \quad U_T = 0. \tag{A.1}$$

Then, U_t is expressed as:

$$U_t = E_t \left[\int_t^T \left(g(c_s, U_s) - \frac{1}{2} A(U_s) ||\sigma_U(s)||^2 \right) ds \right] \tag{A.2}$$

where $c \in \mathscr{C}$ denotes a consumption process and $\mathscr{C}$ is the set of strictly positive consumption processes. $E_t[\cdot]$ denotes a conditional expectation operator given time-t information $\mathscr{F}_t$, $\sigma_U(s)$ denotes a volatility coefficient of the utility process, $A(U_s)$ is a measure of comparative risk aversion over the continuation utility, and the function g captures the preferences over the deterministic consumption paths. The stochastic differential utility, U_t, is essentially characterized by the pair (g, A) and $\sigma_U(t)$. U_0 is the initial value of the utility process U_t.

Set a function $\varphi(\cdot)$ to be twice continuously differentiable and strictly increasing. Define $V_t \triangleq \varphi(U_t)$. Note that U_t and V_t are ordinally equivalent. Using Ito's formula, we obtain:

$$dV_t = \varphi'(U_t) dU_t + \frac{1}{2} \varphi''(U_t) ||\sigma_U(t)||^2 dt \tag{A.3}$$

$$= \left[-\varphi'(U_t) g(c_t, U_t) + \frac{1}{2} \left(\varphi'(U_t) A(U_t) + \varphi''(U_t) \right) ||\sigma_U(t)||^2 \right] dt + \varphi'(U_t) \sigma_U(t) dB_t. \tag{A.4}$$

Therefore, if the function $\varphi(x)$ satisfies an ordinary differential equation:

$$\varphi'(x) A(x) + \varphi''(x) = 0, \tag{A.5}$$

then we obtain a utility process in the following form:

$$V_t = E_t \left[\int_t^T f(c_u, V_u)\, du \right], \tag{A.6}$$

with $f(c_t, V_t) \triangleq \varphi'(\varphi^{-1}(V_t)) g(c_t, \varphi^{-1}(V_t))$.

The ordinally equivalent utility V_t, induced by the converter φ that satisfies Eq.(A.5), is called a normalized stochastic differential utility. Also, the aggregator $f(c_t, V_t)$ is called a normalized aggregator. From Schroder and Skiadas (1999), the stochastic utility function V_0 is concave if the aggregator f is jointly concave in its consumption and utility arguments.

Now, we specify the utility form as a continuous-time version of Kreps and Porteus (1978) with two pairs of (g, A) and $\sigma_U(t)$ in Eq.(A.2):

$$g^{DE}(c,x) \triangleq \beta\log(c/x)x,\ A^{DE}(x) \triangleq \gamma/x,\ \sigma_U^{DE}(t) \triangleq \sigma_{U^{DE}}(t)U_t \tag{A.7}$$

$$g^{SS}(c,x) \triangleq \log c - \beta x,\ A^{SS}(x) \triangleq -\alpha,\ \sigma_U^{SS}(t) \triangleq \sigma_{U^{SS}}(t) \tag{A.8}$$

where $\beta \in (0,\infty),\ \delta \in [0,\infty)$ and $0 < \gamma \neq 1$.

Furthermore, consider the following converters $\varphi(x)$ that are used as the previously explained transformation of the utility function :

$$\varphi^{DE}(x) \triangleq \frac{x^{1-\gamma}-1}{1-\gamma} \tag{A.9}$$

$$\varphi^{SS}(x) \triangleq \frac{e^{\alpha x}-1}{\alpha}. \tag{A.10}$$

Combining the pairs (g, A) with φ respectively, we obtain normalized aggregators f in Eq.(2.9) and Eq.(2.10) (with $\alpha = 1-\gamma$ for the *DE aggregator*).

For above two pairs in Eq.(A.7) and Eq.(A.8), by using Proposition 20 in Skiadas (2007) and Theorem A1 in Schroder and Skiadas (1999), we have the following lemma:

Lemma A.1 *For both the DE aggregator and the SS aggregator, $\gamma - 1 = -\alpha$ is a measure of comparative risk aversion relative to time-separable log utility.*

In other words, when α increases, the risk aversion decreases in Duffie and Epstein (1992)'s sense.
Proof
Due to Skiadas (2007)'s Proposition 20, the following holds: Suppose U^i, $i = 1, 2$, are characterized by the aggregators $(g(c,x), A^i(x))$ in Eq.(A.2) which are identical among $i = 1, 2$ except $\alpha^1 \leq \alpha^2$, then U_0^1 is more risk-averse than U_0^2. □

To interpret $\alpha = 1-\gamma$ intuitively, define another ordinally equivalent utility processes with respect to the *DE aggregator* as follows:

$$\tilde{U}_t^{DE} \triangleq \ln U_t^{DE}. \tag{A.11}$$

The ordinary utility has the following form:

$$\tilde{U}_t^{DE} = E_t\left[\int_t^T e^{-\beta(s-t)}\left(\beta\log c_s + \frac{1-\gamma}{2}||\sigma_{U^{DE}}(s)||^2\right)ds\right]. \tag{A.12}$$

On the other hand, with regard to the *SS aggregator*, from Eq.(A.8) the utility process in Eq.(A.2) is expressed as:

$$U_t^{SS} = E_t\left[\int_t^T e^{-\beta(s-t)}\left(\log c_s + \frac{\alpha}{2}||\sigma_{U^{SS}}(s)||^2\right)ds\right]. \tag{A.13}$$

From Eq.(A.12) and Eq.(A.13), the parameter $\alpha = 1-\gamma$ does not influence the preference for the deterministic consumption but the preference for the uncertainty term of the future continuation utility. Therefore, $\alpha < 0$ means a penalty for continuation-utility uncertainty, whereas $\alpha > 0$ means a reward for it.

Finally, examine the conditions for the concavity of V_0 for each aggregator:

Lemma A.2 *For the DE aggregator, $\alpha > 0$ implies concavity of the utility function V_0, while conditions of $\beta \geq 0$ and $\beta \geq \alpha$ provide concavity of V_0 for the SS aggregator.*

Proof
(the *DE aggregator*:) Joint concavity of f on the domain $\{(c, v) : c > 0, 1 + \alpha v > 0\}$ can be confirmed by checking $f_{cc} < 0$ and $f_{cc} f_{vv} \geq f_{cv}^2$. Then, by Lemma 1 in Schroder and Skiadas (1999), the concavity of V_0 is also assured.
(the *SS aggregator*:) For the case $0 \leq \alpha \leq \beta$, the procedure is the same with the *DE aggregator*. Consider the case of $\alpha < 0$, consider the following: From Theorem A1 in Schroder and Skiadas (1999), U_0 defined by Eq.(A.2) is convex in $\log c$ and thus αU_0 is concave in $\log c$, which immediately means αU_0 is concave in c. On the other hand, since $\alpha < 0$, the function $(e^x - 1)/\alpha$ is concave in x thus $V_0 = (e^{\alpha U_0} - 1)/\alpha$ is concave in αU_0. Combining the two results shows V_0 is concave in c. Moreover, it can be confirmed that with converter $\varphi^{SS}(x) \triangleq (e^{\alpha x} - 1)/\alpha$, the normalized aggregator f is the one in Eq.(2.10). □

A.2 Note on characterization of the utility process $V_t = J(t, X_t)$

Given Eq.(2.1) - Eq.(2.3), with $c_t = e_t$, the functional form of the utility process V_t is obtained as $V_t = J(t, X_t)$ that is the solution to a partial differential equation with $J(T, X_T) = 0$:

Lemma A.3 *The following partial differential equation characterize the utility process $V_t = J(t, X_t)$:*

$$0 = f(e_t, J_t) + J_t + J_x b + \frac{1}{2} a' J_{xx} a. \tag{A.14}$$

Proof
From Eq.(2.1), $V_t + \int_0^t f(c_u, V_u)\, du$ is a martingale:

$$V_t + \int_0^t f(c_u, V_u)\, du = E_t \left[\int_0^T f(c_u, V_u)\, du \right]. \tag{A.15}$$

Since the drift of the process of $V_t + \int_0^t f(c_u, V_u)\, du$ must be zero, an application of Ito's formula to $V_t = J(t, X_t)$ leads to Eq.(A.14). □

Furthermore, the utility process $V_t = J(t, X_t)$ turns out to be

$$dJ_t = -f(e_t, J_t)dt + J_x a\, dB_t. \tag{A.16}$$

A.3 Note on the pricing kernel and the market price of risk

The pricing kernel π_t is expressed by using the instantaneous riskless rate r_t and the market price of risk λ_t as:

$$\pi_t \triangleq \exp\left\{ -\int_0^t r_u\, du \right\} \exp\left\{ -\int_0^t \lambda_u^\top\, dB_u - \frac{1}{2} \int_0^t ||\lambda_u||^2\, du \right\}, \tag{A.17}$$

and then the pricing kernel process π_t is expressed as follows:

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \lambda_t^\top dB_t. \tag{A.18}$$

Then, from Eq.(2.4) and Eq.(A.18), the market price of risk is specified as follows:

Lemma A.4 *The market price of risk, λ_t, is given by*

$$\lambda_t = \sigma_e(t)\left(-\frac{ef^*_{cc}}{f^*_c}\right) + \sigma_J(t)\left(-\frac{f^*_{cv}}{f^*_c}\right). \tag{A.19}$$

For notational convenience, we may use $f^* \triangleq f(e,J)$, $f^*_c \triangleq f_c(e,J)$ and $f^*_v \triangleq f_v(e,J)$ in an abbreviated form. That is, the superscript $*$ of f and its partial derivatives denotes that they are evaluated at equilibrium values (that is, $c = e$ and $v = J$).

Proof of Lemma A.4:

An application of Ito's lemma to π_t in Eq.(2.4) shows:

$$\frac{d\pi_t}{\pi_t} = f_v(e_t, J_t)dt + \frac{df_c(e_t, J_t)}{f_c(e_t, J_t)}. \tag{A.20}$$

Comparing Eq.(A.18) with Eq.(A.20), we see that λ_t must be the diffusion coefficient of $-\frac{df_c}{f_c}$. Then an application of Ito's formula to f_c leads to Eq.(A.19). □

Note that the market price of risk is used to change the physical measure P into a risk neutral measure Q.

To obtain an intuitive understanding of Eq.(A.19), look at the following example. Consider a time-separable aggregator in Eq.(2.7):

$$f^s(c,v) = \beta(u(c) - v), \tag{A.21}$$

where $u(c)$ is a function satisfying $u'(\cdot) > 0$ and $u''(\cdot) < 0$.

Under the time-separable aggregator in Eq.(2.7), the second term on the right hand side of Eq.(A.19) is 0 since $f^*_{cv} = 0$. This term is included in the time non-separable utility, but not in the time-separable utility.

Under the aggregator Eq.(2.7), the pricing kernel is reduced to

$$\pi_t = e^{-\beta t}\frac{u'(e_t)}{u'(e_0)} \tag{A.22}$$

which is the typical form in standard continuous-time asset pricing models with time-separable utility. That is, when the agent has a time-separable utility, she evaluates the risk from time 0 to time t based only on the change in consumption $\Delta c = c_t - c_0$, not on the change in the expectation on consumption ahead of time t. On the other hand, a time non-separable utility depends on the change in the expectation on consumption ahead of time t through the exponential part of Eq.(2.4).

A.4 Note on the DE aggregator with general intertemporal elasticity of substitution

Replace g in Eq.(A.7) by the following:

$$\tilde{g}^{DE}(c,x) \;\; \triangleq \;\; \beta\frac{(c/x)^{1-\delta} - 1}{1-\delta}x \tag{A.23}$$

Then, with φ^{DE} in Eq.(A.9), we obtain a normalized aggregator $\tilde{f}^{DE}$, called the general *DE aggregator*, as follows:

$$\tilde{f}^{DE}(c,v) = \beta \frac{1+(1-\gamma)v}{1-\delta}\left[\left(\frac{c}{[1+(1-\gamma)v]^{1/(1-\gamma)}}\right)^{1-\delta} - 1\right]. \tag{A.24}$$

Note that we obtain a time additive utility when $\gamma = \delta$.

Under the general *DE aggregator*, the following lemma holds:

Lemma A.5 *Under the time non-separable aggregator $\tilde{f}^{DE}$ in Eq.*(A.24)*, the instantaneous riskless rate r_t and the market price of risk λ_t turn out to be:*

$$r_t = \beta + \delta\mu_e(t) - \frac{\delta}{2}||\sigma_e(t)||^2 - \frac{1}{2}||\lambda_t||^2 + \frac{\alpha(1-\gamma)}{2}\left|\left|\frac{\sigma_V}{1+(1-\gamma)V}\right|\right|^2 \tag{A.25}$$

$$\lambda_t = \delta\sigma_e(t) - \alpha\frac{\sigma_J(t)}{1+(1-\gamma)J}. \tag{A.26}$$

Proof

By specifying $\tilde{f}^{DE}$ as Eq.(A.24) and substituting it into Eq.(2.6), we obtain Eq.(A.25). To be more precise, we need to identify f_v^* and $\frac{\mathscr{D}f_c^*}{f_c^*}$. From Eq.(A.24) and $\alpha = \delta - \gamma$, we have

$$f_v(c,v) = \beta\frac{\alpha}{1-\delta}c^{1-\delta}\left[1+(1-\gamma)v\right]^{-(1-\delta)/(1-\gamma)} - \beta\frac{1-\gamma}{1-\delta}, \tag{A.27}$$

$$f_c(c,v) = \beta c^{-\delta}\left[1+(1-\gamma)v\right]^{(\delta-\gamma)/(1-\gamma)}. \tag{A.28}$$

Applying Ito's formula to f_c^*, we have

$$\frac{df_c^*}{f_c^*} = \frac{f_{cc}^*}{f_c^*}de + \frac{f_{cv}^*}{f_c^*}dJ + \frac{1}{2}\left(\frac{f_{ccc}^*}{f_c^*}(de)^2 + 2\frac{f_{ccv}^*}{f_c^*}(de)(dJ) + \frac{f_{cvv}^*}{f_c^*}(dJ^2)\right). \tag{A.29}$$

Hence, we have r_t from Eq.(2.6). On the other hand, substituting $-\frac{f_{cc}^*}{f_c^*} = \frac{\delta}{e}$ and $-\frac{f_{cv}^*}{f_c^*} = -\frac{\alpha}{1+(1-\gamma)J}$ into Eq.(A.19), we obtain Eq.(A.26). □

A.5 Proof of Lemma 2.1

Comparison between Eq.(A.18) and Eq.(A.20) leads to Eq.(2.6). □

A.6 Proof of Proposition 3.1

First, consider the time-separable utility. Note that, since $n = 1$, σ_e is a scalar.

The following lemma states the equilibrium characteristic of the term structure:

Lemma A.6 *Under the aggregator in Eq.*(2.7)*, the equilibrium interest rate and the market price of risk is*

$$r^s = \beta + \frac{-eu''(e)}{u'(e)}(\nu - \beta) + \frac{1}{2}\frac{-e^2u'''(e)}{u'(e)}\sigma_e^2 \tag{A.30}$$

$$\lambda^s = \frac{-eu''(e)}{u'(e)}\sigma_e. \tag{A.31}$$

Especially when we specify $u(c)$ *as:*

$$u(c) \triangleq \frac{c^{1-\gamma}-1}{1-\gamma}, \qquad 0<\gamma\neq 1, \tag{A.32}$$

Eq.(A.30) *and Eq.*(A.31) *are*

$$r^s = (1-\gamma)\beta + \gamma\nu - \frac{1}{2}\gamma(\gamma+1)\sigma_e^2 \tag{A.33}$$

$$\lambda^s = \gamma\sigma_e \tag{A.34}$$

Proof

We apply Lemma 2.1 and Lemma A.4 to the aggregator f^s in Eq.(2.7). Using $f_c = \beta u'(c)$, Eq.(A.29) becomes

$$\frac{df_c^*}{f_c^*} = \frac{eu''(e)}{u'(e)}\left(\frac{de}{e}\right) + \frac{1}{2}\frac{e^2u'''(e)}{u'(e)}\left(\frac{de}{e}\right)^2 \tag{A.35}$$

$$= \frac{eu''(e)}{u'(e)}((\nu-\beta)dt + \sigma_e dB_t) + \frac{1}{2}\frac{e^2u'''(e)}{u'(e)}\sigma_e^2 dt. \tag{A.36}$$

Substituting $f_v^* = \beta$ and $\frac{\mathscr{D}f_c^*}{f_c^*}$ into Lemma 2.1, we obtain r_t^s as in Eq.(A.30). For the market price of risk, substituting $-\frac{ef_{cc}^*}{f_c^*} = -\frac{eu''(e)}{u'(e)}$ and $f_{cv}^* = 0$ into Eq.(A.19), we obtain Eq.(A.31). □

Then, consider the time non-separable aggregator in Eq.(2.9):

$$f^{ns}(c,v) = \beta(1+\alpha v)\left[\log c - \frac{\log(1+\alpha v)}{\alpha}\right]. \tag{A.37}$$

The equilibrium features of the term structure are summarized by the following lemma:

Lemma A.7

$$r_t^{ns} = \nu - \sigma_e^2 + \alpha q(t)\sigma_e^2 = \nu - \sigma_e\lambda_t^{ns} \tag{A.38}$$

$$\lambda_t^{ns} = \sigma_e - \alpha q(t)\sigma_e, \tag{A.39}$$

where $q(t) \triangleq 1 - e^{-\beta(T-t)}$. *Moreover, especially when* $T \to \infty$*,*

$$r^{ns} = \nu - \sigma_e^2 + \alpha\sigma_e^2 \tag{A.40}$$

$$\lambda^{ns} = \sigma_e - \alpha\sigma_e, \tag{A.41}$$

Proof

We apply Eq.(2.6) and Eq.(A.19) to the aggregator f^{ns} in Eq.(A.37). In this case, we need the optimized utility process J to obtain $\frac{\mathscr{D}f_c^*}{f_c^*}$ in Lemma 2.1 and σ_J in Lemma A.4.

First,

$$f_c(c,v) = \frac{\beta(1+\alpha v)}{c} \tag{A.42}$$

$$f_v(c,v) = \alpha\beta\left[\log c - \frac{1}{\alpha}\log(1+\alpha v)\right] - \beta. \tag{A.43}$$

From Eq.(A.16), the process J evolves based on:

$$dJ = -f^{ns}(e,J)dt + J_e e\sigma_e dB. \tag{A.44}$$

Next, applying Ito's formula to f_c^*, we have

$$\begin{aligned}\frac{df_c^*}{f_c^*} &= -\frac{de}{e} + \alpha\frac{dJ}{1+\alpha J} + \left[\left(\frac{de}{e}\right)^2 - \alpha\left(\frac{de}{e}\right)\left(\frac{dJ}{1+\alpha J}\right)\right] &\text{(A.45)}\\ &= -\frac{de}{e} - \alpha\frac{f^*}{1+\alpha J}dt + \alpha\frac{J_e e}{1+\alpha J}\sigma_e dB + \left[\left(\frac{de}{e}\right)^2 - \alpha\left(\frac{de}{e}\right)\left(\frac{J_e e}{1+\alpha J}\sigma_e dB\right)\right]. &\text{(A.46)}\end{aligned}$$

By Eq.(2.6),

$$r_t^{ns} = \nu - \sigma_e^2 + \alpha\frac{J_e e}{1+\alpha J}\sigma_e^2, \tag{A.47}$$

and by Eq.(A.19),

$$\lambda_t^{ns} = \sigma_e - \alpha\frac{J_e e}{1+\alpha J}\sigma_e. \tag{A.48}$$

Thus, all we need is to obtain the explicit expression of $\dfrac{J_e e}{1+\alpha J}$. To do this, we have to solve PDE in Eq.(A.14). With $c = e$, the PDE is reduced to:

$$0 = f^{ns}(e,J) + J_t + J_e e\mu_e + \frac{1}{2}J_{ee}e^2\sigma_e^2 \tag{A.49}$$

with the terminal condition $J(T,e) = 0$. Recall that $\mu_e = \nu - \beta$.

We conjecture the solution of J as:

$$\frac{\log(1+\alpha J)}{\alpha} = q(t)\log e_t + n(t) \tag{A.50}$$

where $q(t)$ and $n(t)$ are time-dependent, deterministic functions. Substituting the conjectured form Eq.(A.50) of J into Eq.(A.49), we have:

$$0 = \beta[\log e_t - (q(t)\log e_t + n(t))] + q'(t)\log e_t + n'(t) + q(t)\mu_e + \frac{1}{2}(\alpha q(t)^2 - q(t))\sigma_e^2. \tag{A.51}$$

Hence,

$$\beta(1 - q(t)) + q'(t) = 0, \tag{A.52}$$

$$-\beta n(t) + n'(t) + q(t)\mu_e + \frac{1}{2}(\alpha q(t)^2 - q(t))\sigma_e^2 = 0. \tag{A.53}$$

By solving these equations, we obtain

$$q(t) = 1 - e^{-\beta(T-t)}, \tag{A.54}$$

$$n(t) = \int_t^T e^{-\beta u}x(u)\,du \tag{A.55}$$

where $x(t) \triangleq \mu_e q(t) + \frac{\sigma_e^2}{2}(\alpha q(t)^2 - q(t))$.

By differentiating Eq.(A.50) with respect to e_t, we obtain $\frac{J_e e}{1+\alpha J} = q(t)$. Substituting this into Eq.(A.47) and Eq.(A.48), we obtain Eq.(A.38) and Eq.(A.39). □

A.7 Proof of Theorem 3.1

First, consider the time-separable case. The following lemma characterizes the equilibrium term structure in this economy:

Lemma A.8

$$
\begin{aligned}
r_t^s &= (1-\gamma)\beta + \gamma\nu_t - \frac{1}{2}\gamma(\gamma+1)||\sigma_e||^2 && \text{(A.56)}\\
\lambda^s &= \gamma\sigma_e && \text{(A.57)}\\
R^s(t,s) &= r_t^s + \gamma\left((\bar{\nu}-\nu_t) - \gamma\frac{\sigma_\nu^\top\sigma_e}{k}\right)\left(1-\frac{1-e^{-k(s-t)}}{k(s-t)}\right)\\
&- \gamma^2\frac{||\sigma_\nu||^2}{2k^2}\left(1-2\frac{1-e^{-k(s-t)}}{k(s-t)}+\frac{1-e^{-2k(s-t)}}{2k(s-t)}\right) && \text{(A.58)}
\end{aligned}
$$

Proof
We apply Lemma 2.1 and Lemma A.4 to the aggregator f^s in Eq.(2.7). From Eq.(A.35), we have

$$\frac{df_c^*}{f_c^*} = \frac{eu''(e)}{u'(e)}\left((\nu_t-\beta)dt + \sigma_e^\top dB\right) + \frac{1}{2}\frac{e^2u'''(e)}{u'(e)}||\sigma_e||^2dt. \tag{A.59}$$

By substituting this into Lemma 2.1 and Lemma A.4, the equilibrium instantaneous spot rate and the market price of risk are

$$
\begin{aligned}
r_t^s &= \beta + \frac{-eu''(e)}{u'(e)}(\nu_t-\beta) - \frac{1}{2}\frac{e^2u'''(e)}{u'(e)}||\sigma_e||^2 && \text{(A.60)}\\
\lambda_t^s &= \frac{-eu''(e)}{u'(e)}\sigma_e. && \text{(A.61)}
\end{aligned}
$$

Especially when we specify $u(c)$ as

$$u(c) \triangleq \frac{c^{1-\gamma}-1}{1-\gamma}, \qquad 0<\gamma\neq 1, \tag{A.62}$$

we have Eq.(A.56) and Eq.(A.57).

Next, to obtain the equilibrium spot yield, we need the equilibrium price of zero coupon bonds. For any zero coupon bonds maturing at time s, we have the following pricing equation:

$$P(t,s) = E_t^Q\left[\exp\left\{-\int_t^s r_u du\right\}\right]. \tag{A.63}$$

To evaluate the expectation in Eq.(A.63), we need the risk neutral process of r_t^s.

From Maruyama-Girsanov's theorem, the stochastic process, $\tilde{B}_t$, defined as:

$$\tilde{B}_t \triangleq B_t + \int_0^t \lambda_u\, du, \tag{A.64}$$

is a Brownian motion under risk neutral measure Q.

Therefore substituting Eq.(A.57) into Eq.(A.64), we have

$$
\begin{aligned}
dB_t &= -\gamma\sigma_e dt + d\tilde{B}_t, && \text{(A.65)}\\
d\nu_t &= [k(\bar{\nu}-\nu_t) - \gamma\sigma_\nu^\top\sigma_e]dt + \sigma_\nu^\top d\tilde{B}_t. && \text{(A.66)}
\end{aligned}
$$

Substituting Eq.(A.56) and Eq.(A.66) into Eq.(A.63), we obtain the following equation:

$$
\begin{aligned}
P^s(t,s) &= \exp\left\{-(s-t)\left[(1-\gamma)\beta - \frac{1}{2}\gamma(\gamma+1)||\sigma_e||^2\right]\right\} E_t^Q\left[\exp\left\{-\gamma\int_t^s \nu_u\,du\right\}\right] \quad \text{(A.67)}\\
&= \exp\Bigg\{-(s-t)r_t - \gamma\frac{k(\bar{\nu}-\nu_t)-\gamma\sigma_\nu^\top\sigma_e}{k}\left[(s-t)-\frac{1-e^{-k(s-t)}}{k}\right]\\
&\qquad +\gamma^2\frac{||\sigma_\nu||^2}{2k^2}\left[(s-t)-2\frac{1-e^{-k(s-t)}}{k}+\frac{1-e^{-2k(s-t)}}{2k}\right]\Bigg\}. \quad \text{(A.68)}
\end{aligned}
$$

Applying the definition $R(t,s) \triangleq -\dfrac{\log P(t,s)}{s-t}$ into Eq.(A.68), we obtain Eq.(A.58). □

Finally, consider the time-nonseparable case. Our stochastic mean economy under the time-nonseparable utility provides the following equilibrium term structure:

Lemma A.9

$$
\begin{aligned}
r_t^{ns} &= \nu_t - ||\sigma_e||^2 + \alpha\left(q(t)||\sigma_e||^2 + m_t\sigma_e^\top\sigma_\nu\right) = \nu_t - \sigma_e^\top\lambda_t^{ns} && \text{(A.69)}\\
\lambda_t^{ns} &= \sigma_e - \alpha\left(q(t)\sigma_e + m_t\sigma_\nu\right) && \text{(A.70)}\\
R^{ns}(t,s) &= r_t^{ns} - \sigma_e^\top\left(\lambda_t^{ns} - \frac{1}{s-t}\int_t^s \lambda_u^{ns}\,du\right)\\
&+ \frac{k[\bar{\nu}-\nu_t]-\sigma_\nu^\top\sigma_e}{k}\left(1-\frac{1-e^{-k(s-t)}}{k(s-t)}\right)\\
&+ \frac{\alpha}{s-t}\left\{\sigma_\nu^\top\sigma_e\int_t^s\left(\int_t^u e^{-k(u-\tau)}q(\tau)\,d\tau\right)du + ||\sigma_\nu||^2\int_t^s\left(\int_t^u e^{-k(u-\tau)}m(\tau)\,d\tau\right)du\right\}\\
&- \frac{||\sigma_\nu||^2}{2k^2}\left(1-2\frac{1-e^{-k(s-t)}}{k(s-t)}+\frac{1-e^{-2k(s-t)}}{2k(s-t)}\right), && \text{(A.71)}
\end{aligned}
$$

Especially when $T\to\infty$ the equilibrium term structure is

$$
\begin{aligned}
r_t^{ns} &= \nu_t - ||\sigma_e||^2 + \alpha\left(||\sigma_e||^2 + \frac{\sigma_e^\top\sigma_\nu}{k+\beta}\right) && \text{(A.72)}\\
\lambda^{ns} &= \sigma_e - \alpha(\sigma_e + \frac{\sigma_\nu}{k+\beta}) && \text{(A.73)}\\
R^{ns}(t,s) &= r_t^{ns} + \left([\bar{\nu}-\nu_t] + \frac{1}{k}\left(-\sigma_\nu^\top\sigma_e + \alpha\left(\sigma_\nu^\top\sigma_e + \frac{||\sigma_\nu||^2}{k+\beta}\right)\right)\right)\left(1-\frac{1-e^{-k(s-t)}}{k(s-t)}\right)\\
&- \frac{||\sigma_\nu||^2}{2k^2}\left(1-2\frac{1-e^{-k(s-t)}}{k(s-t)}+\frac{1-e^{-2k(s-t)}}{2k(s-t)}\right). && \text{(A.74)}
\end{aligned}
$$

Proof

In this case, Eq.(A.16), the optimal utility process J becomes

$$dJ = -f^{ns}(e,J)dt + (J_e e\sigma_e + J_\nu\sigma_\nu)^\top dB. \quad \text{(A.75)}$$

Hence, since $\delta = 1$, we have

$$\lambda_t^{ns} = \sigma_e - \alpha\left(\frac{J_e e}{1+\alpha J}\sigma_e + \frac{J_\nu}{1+\alpha J}\sigma_\nu\right). \quad \text{(A.76)}$$

With regard to r_t^{ns},

$$\frac{\mathscr{D} f_c^*}{f_c^*} = -(\nu_t - \beta) - \alpha\beta[\log e - \frac{1}{\alpha}\log(1+\alpha J)] + ||\sigma_e||^2 - \alpha\sigma_e^\top \left(\frac{J_e e}{1+\alpha J}\sigma_e + \frac{J_\nu}{1+\alpha J}\sigma_\nu\right). \quad (A.77)$$

Hence, by Eq.(2.6),

$$r_t^{ns} = \nu_t - ||\sigma_e||^2 + \alpha\sigma_e^\top \left(\frac{J_e e}{1+\alpha J}\sigma_e + \frac{J_\nu}{1+\alpha J}\sigma_\nu\right) = \nu_t - \sigma_e^\top \lambda_t^{ns}. \quad (A.78)$$

Next, to obtain the closed form of $\frac{J_e e}{1+\alpha J}$ and $\frac{J_\nu}{1+\alpha J}$ we solve PDE in Eq.(A.14). With $c = e$, Eq.(3.4) and Eq.(3.5), the PDE becomes

$$\begin{aligned} 0 &= f^{ns}(e,J) + J_t + J_e e(\nu_t - \beta) + J_\nu k(\bar{\nu} - \nu_t) \\ &+ \frac{1}{2}\left[J_{ee}e^2||\sigma_e||^2 + 2J_{e\nu}e\sigma_e^\top\sigma_\nu + J_{\nu\nu}||\sigma_\nu||^2\right]. \end{aligned} \quad (A.79)$$

We conjecture as follows:

$$\frac{\log[1+\alpha J]}{\alpha} = q(t)\log e_t + m(t)\nu_t + n(t), \quad (A.80)$$

where $q(t)$, $m(t)$ and $n(t)$ are time-dependent deterministic functions.

Substituting Eq.(A.80) into Eq.(A.79), we obtain

$$\begin{aligned} 0 &= q'(t)\log e_t + m'(t)\nu_t + n'(t) + q(t)(\nu_t - \beta) + m(t)k(\bar{\nu} - \nu_t) \\ &+ \frac{1}{2}\left[\{\alpha q(t)^2 - q(t)\}||\sigma_e||^2 + 2\alpha q(t)m(t)\sigma_e^\top\sigma_\nu + \alpha m(t)^2||\sigma_\nu||^2\right] \\ &+ \beta\left(\log e_t - [q(t)\log e_t + m(t)\nu_t + n(t)]\right). \end{aligned} \quad (A.81)$$

Solving Eq.(A.79), we have

$$\begin{aligned} q(t) &= 1 - e^{-\beta(T-t)} && (A.82) \\ m(t) &= \frac{1 - e^{-(k+\beta)(T-t)}}{k+\beta} - \frac{e^{-\beta(T-t)} - e^{-(k+\beta)(T-t)}}{k} && (A.83) \\ n(t) &= \int_t^T e^{-\beta(u-t)}x(u)\,du && (A.84) \end{aligned}$$

where

$$\begin{aligned} x(t) &\triangleq k\bar{\nu}m(t) - \beta q(t) \\ &+ \frac{1}{2}\left[\{\alpha q(t)^2 - q(t)\}||\sigma_e||^2 + 2\alpha q(t)m(t)\sigma_e^\top\sigma_\nu + \alpha m(t)^2||\sigma_\nu||^2\right]. \end{aligned} \quad (A.85)$$

Taking the limit as $T \to \infty$, Eq.(A.82) and Eq.(A.83) become:

$$\begin{aligned} q(t) &\to 1 && (A.86) \\ m(t) &\to \frac{1}{k+\beta}. && (A.87) \end{aligned}$$

Differentiating Eq.(A.80) with respect to e_t and ν_t, we obtain $\frac{eJ_e}{1+\alpha J} = q(t)$ and $\frac{J_\nu}{1+\alpha J} = m(t)$. Consequently, we have Eq.(A.69) and Eq.(A.70).

Next, with regard to the spot yield, use Eq.(A.63). We need the risk neutral process of r_t^{ns}. Substituting Eq.(A.70) into Eq.(A.64), we have:

$$dB_t = -\left(\sigma_e - \alpha(q(t)\sigma_e + m(t)\sigma_\nu)\right) dt + d\tilde{B}_t, \tag{A.88}$$

and thus the risk neutral process of ν_t is given by

$$d\nu_t = \left[k(\bar{\nu} - \nu_t) - \sigma_\nu^\top \left(\sigma_e - \alpha(q(t)\sigma_e + m(t)\sigma_\nu)\right)\right] dt + \sigma_\nu^\top d\tilde{B}_t. \tag{A.89}$$

Substituting Eq.(A.69) and Eq.(A.89) into Eq.(A.63), we obtain

$$\begin{aligned} P^{ns}(t,s) &= \exp\Bigg\{-(s-t)\left[r_t^{ns} - \sigma_e^\top\left(\lambda_t^{ns} - \frac{1}{s-t}\int_t^s \lambda_u^{ns}\,du\right)\right] \\ &\quad -\frac{k[\bar{\nu}-\nu_t] - \sigma_\nu^\top\sigma_e}{k}\left((s-t) - \frac{1-e^{-k(s-t)}}{k}\right) \\ &\quad -\alpha\left\{\sigma_\nu^\top\sigma_e\int_t^s\left(\int_t^u e^{-k(u-\tau)}q(\tau)\,d\tau\right)du + ||\sigma_\nu||^2\int_t^s\left(\int_t^u e^{-k(u-\tau)}m(\tau)\,d\tau\right)du\right\} \\ &\quad +\frac{||\sigma_\nu||^2}{2k^2}\left((s-t) - 2\frac{1-e^{-k(s-t)}}{k} + \frac{1-e^{-2k(s-t)}}{2k}\right)\Bigg\}. \end{aligned} \tag{A.90}$$

Especially when $T \to \infty$, the zero coupon price is:

$$\begin{aligned} P^{ns}(t,s) &= \exp\Bigg\{-(s-t)r_t^{ns} - \frac{k[\bar{\nu}-\nu_t] - \sigma_\nu^\top\sigma_e}{k}\left((s-t) - \frac{1-e^{-k(s-t)}}{k}\right) \\ &\quad -\alpha\Bigg\{\sigma_\nu^\top\sigma_e\left((s-t) - \frac{1-e^{-k(s-t)}}{k}\right) + \frac{||\sigma_\nu||^2}{k+\beta}\left((s-t) - \frac{1-e^{-k(s-t)}}{k}\right)\Bigg\} \\ &\quad +\frac{||\sigma_\nu||^2}{2k^2}\left((s-t) - 2\frac{1-e^{-k(s-t)}}{k} + \frac{1-e^{-2k(s-t)}}{2k}\right)\Bigg\}. \end{aligned} \tag{A.91}$$

Applying the definition of $R(t,s)$ to Eq.(A.90) and Eq.(A.91), we have Eq.(A.71) and Eq.(A.74). □